EXISTENCE OF SUPERSYMMETRIC HERMITIAN METRICS WITH TORSION ON NON-KAHLER MANIFOLDS

JI-XIANG FU AND SHING-TUNG YAU

1. Introduction

In their proposed compactification of superstring [6], Candelas, Horowitz, Strominger and Witten took the matric product of a maximal symmetric four dimensional spacetime M with a six dimensional Calabi-Yau vacua X as the ten dimensional spacetime; they identified the Yang-Mills connection with the SU(3) connection of the Calabi-Yau metric and set the dilaton to be a constant. To make this theory compatible with the standard grand unified field theory, Witten [22] and Horava-Witten [12] proposed to use higher rank bundles for strong coupled heterotic string theory so that the gauge groups can be SU(4) or SU(5). Mathematically, this approach relies on Uhlenbeck-Yau's theorem on constructing Hermitian-Yang-Mills connections over stable bundles [20].

In [18], A. Strominger analyzed heterotic superstring background with spacetime sypersymmetry and non-zero torsion by allowing a scalar "warp factor" to multiply the spacetime metric. He considered a ten dimensional spacetime that is the product $M\times X$ of a maximal symmetric four dimensional spacetime M and an internal space X; the metric on $M\times X$ takes the form

$$e^{2D(y)}\begin{pmatrix} g_{ij}(y) & 0\\ 0 & g_{\mu\nu}(x)\end{pmatrix},\qquad x\in X,\quad y\in M;$$

the connection on an auxiliary bundle is Hermitian-Yang-Mills over X:

$$F\wedge\omega^2=0,\quad F^{2,0}=F^{0,2}=0.$$

Here ω is the hermitian form $\omega=\frac{\sqrt{-1}}{2}g_{i\bar j}dz^i\wedge d\bar z^j$. In this system, the physical relevant quantities are

$$h=\frac{\sqrt{-1}}{2}(\bar\partial-\partial)\omega,$$

$$\phi=\frac{1}{8}\log\|\Omega\|+\phi_0,$$

and

$$g^0_{ij}=e^{2\phi_0}\|\Omega\|^{\frac14}g_{ij},$$

for a constant ϕ_0. The spacetime supersymmetry forces $D(y)$ to be the dilaton field.

In order for such ansatze to provide a supersymmetric configuration, one introduces a Majorana-Weyl spinor ϵ so that

$$\delta\phi^0_j=\nabla^0_j\epsilon^0+\frac{1}{48}e^{2\phi}\big(\gamma^0_jH^0-12h^0_j\big)\epsilon^0=0,$$

$$\delta\lambda^0=\nabla^0\phi\epsilon^0+\frac{1}{24}e^{2\phi}h^0\epsilon^0=0,$$

$$\delta\chi^0=e^{\phi}F_{ij}\Gamma^{0ij}\epsilon^0=0,$$

where ψ^0 is the gravitano, λ^0 is the dilatino, χ^0 is the gluino, ϕ is the dilaton and h is the Kalb-Ramond filed strength obeying[1]

$$dh = \alpha'(\mathrm{tr}F \wedge F - \mathrm{tr}R \wedge R),$$

where $\alpha' > 0$. (For details of this discussion, please consult [18, 19].) By suitably transforming these quantities, Strominger showed that in order to achieve space-time supersymmetry the internal six manifold X must be a complex manifold with a non-vanishing holomorphic three form Ω; the Hermitian form ω must obey

$$\sqrt{-1}\partial\bar{\partial}\omega = \alpha'(\mathrm{tr}F \wedge F - \mathrm{tr}R \wedge R)$$

and[2]

$$d^*\omega = \sqrt{-1}(\bar{\partial} - \partial)\log \|\Omega\|_\omega.$$

Accordingly, he proposed to solve the system

$$F_H \wedge \omega^2 = 0; \tag{1.1}$$

$$F_H^{2,0} = F_H^{0,2} = 0; \tag{1.2}$$

$$\sqrt{-1}\partial\bar{\partial}\omega = \alpha'(\mathrm{tr}F_H \wedge F_H - \mathrm{tr}R \wedge R); \tag{1.3}$$

$$d^*\omega = \sqrt{-1}(\bar{\partial} - \partial)\ln \|\Omega\|_\omega \tag{1.4}$$

that are solutions of superstring with torsion that allows non-trivial dilaton field and Yang-Mills field. Here ω is the Hermitian form and R is the curvature tensor of the Hermitian metric ω; H is the Hermitian metric and F is its curvature of a vector bundle E; the tr is the trace of the endomorphism bundle of either E or TX.

In [15], Li and Yau have proven the following useful:

Lemma 1. *The equation (1.4) is equivalent to*

$$d(\| \Omega \|_\omega \omega^2) = 0. \tag{1.5}$$

In their paper, Li and Yau have given the first irreducible non-singular solution of the supersymmetric system of Strominger for $U(4)$ and $U(5)$ principle bundle. They obtain their solutions by perturbing around the Calabi-Yau vacua paired with the gauge field that is the tangent connection.

It was speculated by M. Reid that all Calabi-Yau manifolds can be deformed to each other through conifold transition. To achieve this goal, it is inevitable that we must work with non-Kahler manifolds.

The most common examples of non-Kaehler manifolds X are some T^2 bundles over Calabi-Yau varieties [3, 4, 7, 9, 11, 13]. Because internal six manifold X is a complex manifold with a non-vanishing holomorphic three form Ω, at first we may consider the T^2-bundle (X, ω, Ω) over complex surface (S, ω_S, Ω_S) with non-vanishing holomorphic 2-form Ω_S. According to the classification of complex surfaces by Enriques and Kodaira, such surfaces include K3 surface and complex torus (Calabi-Yau) and Kodaira surface (non-Kahler). If (X, ω, Ω) satisfies the Strominger's equation (1.4), then by Lemma 1, $d(\| \Omega \|_\omega \omega^2) = 0$. If

[1]The curvature F of vector bundle E in ref.[18] is real, i.e., $c_1(E) = \frac{F}{2\pi}$. But we are used to take the curvature F satisfying $c_1(E) = \frac{\sqrt{-1}}{2\pi}F$.

[2]See eq. (56) of ref.[19], which corrects eq. (2.30) of ref.[18] by a minus sign.

we let $\omega' = \| \Omega \|_{\omega}^{\frac{1}{2}} \omega$, then $d\omega'^2 = 0$, i.e., ω' is a balanced metric [17]. Balanced metric is a very interesting concept. This was studied extensively by Michelson. For example, Michelson proved that the balanced condition is preserved under the proper holomorphic submersion. Note that Alessandrini and Bassanelli [1] proved that this condition is also preserved under the modofications. Now X is balanced and holomorphic submersion π from X to complex surface S is proper, so S is also balanced (actually $\pi_* \omega'^2$ is the balanced metric, see proposition 1.9 in [17]). Note that when the dimension of complex manifold is 2, the conditions of being balanced and Kaehler concide. So S is Kaehler. Then there is no solution to Strominger's equation on T^2 bundles over Kodaira surface and we should only consider the case of K3 surface and complex torus.

Up to now, only known example of the solution to Strominger's system on non-Kahler manifold is given by Cardoso, Curio, Dall'Agata and Lust in [7]. By calculating the curvature, they have given the reducible solution on the Iwasawa manifold which is some T^2 bundle over complex torus.

On the other hand, when K. Becker, M. Becker , K. Dasgupta etc. finished their two papers on compactification of heterotic theory on non-Kahler complex manifolds [3, 4], M. Becker and K. Dasgupta in their review paper [5] think that the question of finding stable vector bundle for their manifolds (i.e., some T^2 bundles over K3 surfaces) is a very important one especially because we are no longer allowed to embed the spin connection (i.e., the connection with torsion $\sqrt{-1}(\bar{\partial} - \partial)\omega$) into the gauge connection (hermitian connection). They think that Strominger's equations (1.3) and (1.4) look very restrictive and one might wonder if there exists any solution at all to the equations.

In this paper, we will construct this solution on some torus bundles over K3 surface or complex torus provided by Goldstein and Prokushkin [9]. Let (S, ω_S, Ω_S) be a K3 surface or complex torus with kahler form ω_S and a non-vanishing holomorphic (2,0) form Ω_S. Let ω_1 and ω_2 are anti-self-dual (1,1) forms such that $\frac{\omega_1}{2\pi}$ and $\frac{\omega_2}{2\pi}$ represent integral cohomology classes. Using these two forms, Goldstein and Prokushkin constructed the non-Kahler manifold X such that $\pi : X \to S$ is a holomorphic T^2 fibration over S with hermitian form $\omega_0 = \pi^* \omega_S + \frac{\sqrt{-1}}{2} \theta \wedge \bar{\theta}$ and holomorphic 3 form $\Omega = \Omega_S \wedge \theta$ (The definition of θ see [9] or section 3). Now we construct the superstring as follows.

Let L_1 and L_2 be holomorphic line bundle over S such that their curvatures are $\sqrt{-1}\omega_1$ and $\sqrt{-1}\omega_2$ respectively. Corresponding to these curvature, there exist hermitian metrics h_1 and h_2 on L_1 and L_2. Let $E = L_1 \oplus L_2 \oplus T'S$ and let $H_0 = (h_1, h_2, \omega_S)$. Then $F_{H_0} = \mathrm{diag}(\sqrt{-1}\omega_1, \sqrt{-1}\omega_2, R_{\omega_S})$. Let u be any smooth function on S and let

$$\omega_u = \pi^*(e^u \omega_S) + \frac{\sqrt{-1}}{2} \theta \wedge \bar{\theta}. \tag{1.6}$$

Then $(V = \pi^* E, \pi^* F_{H_0}, X, \omega_u)$ satisfies the Strominger's equation (1.1),(1.2) and (1.4). So we should only need to consider the equation (1.3). Because ω_1 and ω_2 are harmonic, locally write ω_1 and ω_2 as

$$\omega_1 = \bar{\partial} \xi = \bar{\partial}(\xi_1 dz_1 + \xi_2 dz_2)$$

and

$$\omega_2 = \bar{\partial} \zeta = \bar{\partial}(\zeta_1 dz_1 + \zeta_2 dz_2),$$

where (z_1, z_2) is the local coordinate on S. Let

$$A = \begin{pmatrix} \xi_1 + \sqrt{-1}\zeta_1 \\ \xi_2 + \sqrt{-1}\zeta_2 \end{pmatrix}.$$

Using matrix A we can calculate the curvature R_u of metric ω_u and $R_u \wedge R_u$. Let $g = (g_{i\bar{j}})$ if $\omega_S = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}}$. We can prove

Theorem 2. *$(V = \pi^* E, \pi^* F_{H_0}, X, \omega_u)$ is the solution of Strominger's system if and only if the function u of S satisfies the equation*

$$\triangle e^u \cdot \frac{\omega_S^2}{2!} + \partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1})) + \partial\bar{\partial}u \wedge \partial\bar{\partial}u = 0. \tag{1.7}$$

In particular, when $\omega_2 = n\omega_1, n \in \mathbb{Z}$, $(V, \pi^* F_{H_0}, X, \omega_u)$ is the solution to Strominger's system if and only if smooth function u on S satisfies the equation:

$$\triangle \left(e^u + \frac{(1+n^2)}{4} \parallel \omega_1 \parallel_{\omega_S}^2 e^{-u} \right) - 8\frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} = 0. \tag{1.8}$$

Actually we can prove that

$$\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g_S^{-1})$$

is a globally well-defined (1,1)-form on S. In particular, when $\omega_2 = n\omega_1$, $n \in \mathbb{Z}$,

$$\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g_S^{-1}) = \sqrt{-1}\frac{1+n^2}{4} \parallel \omega_1 \parallel_{\omega_S}^2 \omega_S.$$

Let $f = \frac{(1+n^2)}{4} \parallel \omega_1 \parallel^2$. If we let $g'_{i\bar{j}} = (e^u - fe^{-u})g_{i\bar{j}} - 4u_{i\bar{j}}$, then we can rewrite the equation (1.8) as

$$\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = (e^u - fe^{-u})^2 + 2\{(e^u + fe^{-u}) \mid \triangledown u \mid^2 + e^{-u} \triangle f - 2e^{-u} \triangledown u \cdot \triangledown f\}$$

We solve equation (1.7) by the continuity method [23]. We will prove

Theorem 3. *There is an unique solution of equation (1.7) under the elliptic condition $e^u\omega_S + \sqrt{-1}e^{-u}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g_S^{-1}) - 2\sqrt{-1}\partial\bar{\partial}u > 0$ and the normalization $\int_S e^{-u} = A << 1$. If $\omega_2 = n\omega_1$, then*

$$A < \min \left\{ 1, C_1^{-1} \left(\max\{(7^{\frac{1}{3}}, (2C_1)^2, (1 + \sup f), 16(\max R_{i\bar{j}k\bar{l}} + 1)\} \right)^{-\frac{2}{B}} \right\}$$

where C_1 depends only on S (it can be written by P. Li's notation in [16]*) and constant B is*

$$B = \prod_{\beta=1}^{\infty} \left(1 - \frac{1}{2^\beta} \right) > 0$$

Actually we can get the estimate $\inf u \geq -\ln C_1 - \frac{B}{2}\ln A$. So if $A < C_1^{-\frac{2}{B}}$, then $\inf u > 0$. This is important in our estimate.

Fix the solution u of equation (1.7). Then according to theorem 2, we get the reducible solution $(V, F_{\pi^* H_0}, X, \omega_u)$. It can be extended to a family of irreducible solution by perturbing around it. So we follow Li-Yau's method [15] and get the following

Theorem 4. *Let (E, H_0, S, ω_S) be as before. Fix its holomorphic structure D''_0. Then there is a smooth deformation D''_s of (E, D''_0) so that there are Hermitian-Yang-Mills metric H_s on (E, D_s) and smooth function ϕ_s on S such that*

$$\left(V = \pi^* E, \pi^* D''_s, \pi^* H_s, \pi^*(e^{u+\phi_s}\omega_S) + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta} \right)$$

are the irreducible solutions to strominger's system on X and so that $\lim_{s\to 0}\phi_s = 0$ and $\lim_{s\to 0} H_s$ is a regular reducible hermitian Yang-Mills connection on $E = L_1 \oplus L_2 \oplus TS$.

The organization of the paper is as follows: sect. 2 is a review of some results of paper [9]. In sect. 3 we calculate $\mathrm{tr} R \wedge R$. Then we can construct the reducible solution (Theorem 2) in sect. 4 and get irreducible solution (theorem 4) in sect. 5. From sect. 6, we solve the equation (1.7) (Theorem 3) by continuity method. At first, we prove the openness in sect. 6. We do the estimates up to third order from sect. 7 to sect. 10. In order to write everything down clearly and easily, we do estimates only to equation (1.8). Then we summarize the estimates in sect. 11 to get the closeness for equation (1.8). Finally, in sect. 12, we explain why we can easily generalize our estimates for equation (1.8) to estimates for equation (1.7).

Acknowledgement. J.X. Fu would like to thank Professor Jun Li for interesting discussions and helps. The Proposition 15 is due to Jun Li. J.X. Fu is supported by NSFC grant 10471026. S.T. Yau is supported partially by NSF grants DMS-0306600 and DMS-0074329.

2. Geometric Modules

In this section, we take Goldestein and Prokushkin's geometric model for complex non-Kahler manifolds with $SU(3)$ structure [9]. We organize their some results as the following:

Theorem 5. [9] *Let (S, ω_S, Ω_S) be a Calabi-Yau 2-fold with a non-vanishing holomorphic $(2,0)-$form Ω_S. Let ω_1 and ω_2 be closed 2-forms on S satisfying the following conditions:*
(1). ω_1 and ω_2 are anti-self dual $(1,1)$-forms, $*\omega_1 = -\omega_1, *\omega_2 = -\omega_2$, which are equivalent to

$$\omega_1 \wedge \omega_S = 0, \qquad \omega_2 \wedge \omega_S = 0. \tag{2.1}$$

(2). $\frac{\omega_1}{2\pi}$ and $\frac{\omega_2}{2\pi}$ represent integral cohomology classes.
Then there is a hermitian 3-fold X such that $\pi : X \to S$ is a holomorphic T^2-fibration over S such that following holds:
1. For any 1-forms α and β defined on some open subset of S and satisfying $d\alpha = \omega_1$ and $d\beta = \omega_2$ there are local coordinates x and y on X such that $dx + idy$ is a holomorphic form on T^2-fibers and the metric on X has the following form:

$$g_0 = \pi^* g + (dx + \pi^*\alpha)^2 + (dy + \pi^*\beta)^2 \tag{2.2}$$

where g is the Calabi-Yau metric on S corresponding to Kahler form ω_S.
2. X admits a nowhere vanishing holomorphic $(3,0)$-form with unit length:

$$\Omega = ((dx + \pi^*\alpha) + i(dy + \pi^*\beta)) \wedge \pi^*\Omega_S$$

3. If either ω_1 or ω_2 represent a non-trivial cohomological class then X admits no Kahler metric.
4. But X is a balanced manifold [17]. *Actually hermitian form*

$$\omega_0 = \pi^*\omega_S + (dx + \pi^*\alpha) \wedge (dy + \pi^*\beta); \tag{2.3}$$

corresponding to the metric (2.2) is balanced, i.e., $d\omega_0^2 = 0$;
5. Furthermore, for any smooth function u on S, the hermitian metric

$$\omega_u = \pi^*(e^u\omega_S) + (dx + \pi^*\alpha) \wedge (dy + \pi^*\beta)$$

is also balanced.

Goldestein and Prokushkin also have studied the cohomology of this non-Kähler manifold X:

$$h^{1,0}(X) = h^{1,0}(S),$$
$$h^{0,1}(X) = h^{0,1}(S) + 1;$$

In particular

$$h^{0,1}(X) = h^{1,0}(X) + 1.$$

Moreover,

$$\begin{aligned} b_1(X) &= b_1(S) + 1, && \text{when} \;\; \omega_2 = n\omega_1, \\ b_1(X) &= b_1(S), && \text{when} \;\; \omega_2 \neq n\omega_1; \\ b_2(X) &= b_2(S) - 1, && \text{when} \;\; \omega_2 = n\omega_1, \\ b_2(X) &= b_2(S) - 2, && \text{when} \;\; \omega_2 \neq n\omega_1 \end{aligned}$$

and

$$b_3(X) = 0.$$

3. The calculation of $\mathrm{tr} R \wedge R$

In order to calculate the curvature R and $\mathrm{tr}R \wedge R$, we should express the Hermitian metric (2.2) under some basis of holomorphic (1,0) vector fields. So at first we should write down the complex structure on X. Let $\{U, z_j = x_j + \sqrt{-1}y_j, j = 1, 2\}$ be a local coordinate in S. The horizontal lifts of vector fields $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_j}$ which are in the kernel of $dx + \pi^*\alpha$ and $dy + \pi^*\beta$ are

$$X_j = \frac{\partial}{\partial x_j} - \alpha\left(\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x} - \beta\left(\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial y} \quad \text{for} \;\; j = 1, 2,$$

$$Y_j = \frac{\partial}{\partial y_j} - \alpha\left(\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial x} - \beta\left(\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial y} \quad \text{for} \;\; j = 1, 2.$$

Then the complex structure $\tilde{I}$ on X is defined as

$$\begin{aligned} \tilde{I}X_j &= Y_j, \quad \tilde{I}Y_j = -X_j, \quad \text{for} \;\; j = 1, 2, \\ \tilde{I}\frac{\partial}{\partial x} &= \frac{\partial}{\partial y}, \quad \tilde{I}\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}. \end{aligned}$$

Let

$$\begin{aligned} U_j &= X_j - \sqrt{-1}\tilde{I}X_j = X_j - \sqrt{-1}Y_j, \\ U_0 &= \frac{\partial}{\partial x} - \sqrt{-1}\tilde{I}\frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}. \end{aligned}$$

Then $\{U_j, U_0\}$ is the basis of the (1,0) vector fields on X. Under this basis, the metric (2.2) takes the following hermitian matrix:

$$\begin{pmatrix} (g_{i\bar{j}}) & 0 \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

because U_1 and U_2 are in the kernel of $dx + \pi^*\alpha$ and $dy + \pi^*\beta$. Let

$$\theta = dx + \sqrt{-1}dy + \pi^*(\alpha + \sqrt{-1}\beta) \tag{3.2}$$

It's easily checked that $\{\pi^*d\bar{z}_j, \overline{\theta}\}$ annihilates the $\{U_j, U_0\}$ and is the basis of (0,1) forms on X. So $\{\pi^*dz_j, \theta\}$ are (1,0) forms on X. Certainly π^*dz_j are holomorphic (1,0) forms and θ is not. So we should construct another holomorphic (1,0) form on X. Because ω_1 and ω_2 are harmonic forms on S, $\overline{\partial}\omega_1 = \overline{\partial}\omega_2 = 0$. Locally we can find (1,0) forms $\xi = \xi_1 dz_1 + \xi_2 dz_2$

and $\zeta = \zeta_1 dz_1 + \zeta_2 dz_2$ on S , where ξ_i and ζ_j are smooth complex functions on some open set of S, such that $\omega_1 = \overline{\partial}\xi$ and $\omega_2 = \overline{\partial}\zeta$. Let

$$\begin{aligned} \theta_0 &= \theta - \pi^*(\xi + \sqrt{-1}\zeta) \\ &= (dx + \sqrt{-1}dy) + \pi^*(\alpha + \sqrt{-1}\beta) - \pi^*(\xi + \sqrt{-1}\zeta) \end{aligned}$$

We claim that θ_0 is the holomorphic (1,0) form. By our construction, θ_0 is the (1,0) form. So we should only explain that θ_0 is holomorphic. Because θ is a (1,0)-form on X, then $\partial\theta$ is a (2,0)-form. But $d\theta = d(dx + \sqrt{-1}dy + \pi^*(\alpha + \sqrt{-1}\beta)) = \pi^*(\omega_1 + \sqrt{-1}\omega_2)$ is a (1,1) form on X. So

$$\partial\theta = 0 \ \text{ and } \ \overline{\partial}\theta = d\theta = \pi^*(\omega_1 + i\omega_2). \tag{3.3}$$

Thus we have

$$\begin{aligned} \overline{\partial}\theta_0 &= \overline{\partial}\theta - \overline{\partial}\pi^*(\xi + \sqrt{-1}\zeta) \\ &= \pi^*(\omega_1 + \sqrt{-1}\omega_2) - \pi^*(\omega_1 + \sqrt{-1}\omega_2) = 0 \end{aligned}$$

So θ_0 is the holomorphic (1,0) form and $\{\pi^* dz_j, \theta_0\}$ is the basis of holomorphic (1,0) forms on X. Therefore we can construct the basis of holomorphic vector fields, which is dual to the basis of $\{\pi^* dz_j, \theta_0\}$. Let

$$\varphi_j = \xi_j + \sqrt{-1}\zeta_j \quad \text{for} \quad j = 1, 2$$

and

$$\tilde{U}_j = U_j + \varphi_j U_0 \quad \text{for} \quad j = 1, 2$$

Then it's easily checked that $\{\tilde{U}_j, U_0\}$ is dual to $\{\pi^* dz_j, \theta_0\}$ because U_j is in the kernel of θ. So it's the basis of holomorphic (1,0) vector fields. Under this basis, the metric g_0 takes the following hermitian matrix:

$$H_X = \begin{pmatrix} g_{1\bar{1}} + \mid \varphi_1 \mid^2 & g_{1\bar{2}} + \varphi_1 \overline{\varphi}_2 & \varphi_1 \\ g_{2\bar{1}} + \varphi_2 \overline{\varphi}_1 & g_{2\bar{2}} + \mid \varphi_2 \mid^2 & \varphi_2 \\ \overline{\varphi}_1 & \overline{\varphi}_2 & 1 \end{pmatrix} = \begin{pmatrix} g + A \cdot A^* & A \\ A^* & 1 \end{pmatrix} \tag{3.4}$$

where g is the Calabi-Yau metric on S and $A = (\varphi_1, \varphi_2)^t$.

According to Strominger's explain in ref [18], when the manifold is not Kahler, we should take the curvature of Hermitian connection on the holomorphic tangent bundle $T'X$. Using the metric (3.4), we can easily calculate the connection and curvature. By directly calculation, we get the curvature

$$R = \overline{\partial}(\partial H_X \cdot H_X^{-1}) = \begin{pmatrix} R_{1\bar{1}} & R_{1\bar{2}} \\ R_{2\bar{1}} & R_{2\bar{2}} \end{pmatrix}$$

where

$$\begin{aligned} R_{1\bar{1}} &= R_S + \overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) + A \cdot \overline{\partial}(\partial A^* \cdot g^{-1}) \\ R_{1\bar{2}} &= -R_S A + (\partial g \cdot g^{-1}) \wedge \overline{\partial}A - \overline{\partial}A \wedge (\partial A^* \cdot g^{-1})A \\ &\quad -A\overline{\partial}(\partial A^* \cdot g^{-1})A + A(\partial A^* \cdot g^{-1}) \wedge \overline{\partial}A + \overline{\partial}\partial A \\ R_{2\bar{1}} &= \overline{\partial}(\partial A^* \cdot g^{-1}) \\ R_{2\bar{2}} &= -\overline{\partial}(\partial A^* \cdot g^{-1})A + (\partial A^* \cdot g^{-1}) \wedge \overline{\partial}A \end{aligned}$$

and R_S is the curvature of Calabi-Yau metric g on S. It is easily checked that $\mathrm{tr}(\overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) + A \cdot \overline{\partial}(\partial A^* \cdot g^{-1})) - \overline{\partial}(\partial A^* \cdot g^{-1})A + (\partial A^* \cdot g^{-1}) \wedge \overline{\partial}A = 0$. So $\mathrm{tr}R = \pi^* \mathrm{tr}R_S$.

Proposition 6. [10] *The Ricci forms of the hermitian connections on X and S have the relation $trR = \pi^* trR_S$.*

Remark 7. *In the above calculation, we don't use the condition that the metric g on S is Calabi-Yau.*

Next we should calculate the $\mathrm{tr}R \wedge R$.

Proposition 8.

$$\mathrm{tr}R \wedge R = \pi^*(\mathrm{tr}R_S \wedge R_S + 2\mathrm{tr}\partial\overline{\partial}(\overline{\partial}A \wedge \partial A^* \cdot g^{-1})). \tag{3.5}$$

Proof. We take the following trick. Fix the point $p \in S$ and pick A such that $A(p) = 0$, e.g., we can take the gauge transformation to get this point. So when we calculate the $\mathrm{tr}R \wedge R$ at the point p, all terms containing the factor A will vanish. Thus

$$\begin{aligned}
& \mathrm{tr}R \wedge R \\
= \; & \mathrm{tr}R_S \wedge R_S + 2\mathrm{tr}R_S \wedge \overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) \\
& +2\mathrm{tr}\partial g \cdot g^{-1} \wedge \overline{\partial}A \wedge \bar{\partial}(\partial A^* \cdot g^{-1}) + 2\mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) \\
& +\mathrm{tr}\overline{\partial}A \wedge ((\partial A^* \cdot g^{-1}) \wedge \overline{\partial}A \wedge (\partial A^* \cdot g^{-1})) \\
& +((\partial A^* \cdot g^{-1}) \wedge \overline{\partial}A \wedge (\partial A^* \cdot g^{-1})) \wedge \overline{\partial}A \\
= \; & \mathrm{tr}R_S \wedge R_S + 2\mathrm{tr}R_S \wedge \overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) \\
& +2\mathrm{tr}\partial g \cdot g^{-1} \wedge \overline{\partial}A \wedge \bar{\partial}(\partial A^* \cdot g^{-1}) + 2\mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1})
\end{aligned}$$

We finish the proof of this proposition by proof of the following two claims.

Claim 1.

$$\begin{aligned}
\mathrm{tr}\partial\overline{\partial}(\overline{\partial}A \wedge \partial A^* \cdot g^{-1}) \;=\; & \mathrm{tr}R_S \wedge \overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) \\
& +\mathrm{tr}\partial g \cdot g^{-1} \wedge \overline{\partial}A \wedge \bar{\partial}(\partial A^* \cdot g^{-1}) \\
& +\mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1})
\end{aligned}$$

Proof.

$$\begin{aligned}
& \mathrm{tr}\partial\overline{\partial}(\overline{\partial}A \wedge \partial A^* \cdot g^{-1}) \\
= \; & -\mathrm{tr}\partial(\overline{\partial}A \wedge \overline{\partial}(\partial A^* \cdot g^{-1})) \\
= \; & \mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) + \mathrm{tr}\overline{\partial}A \wedge \partial\overline{\partial}(\partial A^* \cdot g^{-1}) \\
= \; & \mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) + \mathrm{tr}\overline{\partial}A \wedge \overline{\partial}(\partial A^* \wedge \partial g^{-1}) \\
= \; & \mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) - \mathrm{tr}\overline{\partial}A \wedge \overline{\partial}(\partial A^* \cdot g^{-1} \wedge \partial g \cdot g^{-1}) \\
= \; & \mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) - \mathrm{tr}\overline{\partial}A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) \wedge \partial g \cdot g^{-1} \\
& +\mathrm{tr}\overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) \wedge \bar{\partial}(\partial g \cdot g^{-1}) \\
= \; & \mathrm{tr}\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) - \mathrm{tr}\overline{\partial}A \wedge \overline{\partial}(\partial A^* \cdot g^{-1}) \wedge \partial g \cdot g^{-1} \\
& +\mathrm{tr}\overline{\partial}A \wedge (\partial A^* \cdot g^{-1}) \wedge R_S \\
= \; & \mathrm{tr}(\overline{\partial}\partial A \wedge \overline{\partial}(\partial A^* \cdot g^{-1})) + \mathrm{tr}(R_S \wedge \overline{\partial}A \wedge \partial A^* \cdot g^{-1}) \\
& +\mathrm{tr}(\partial g \cdot g^{-1} \wedge \overline{\partial}A \wedge \overline{\partial}(\partial A^* \cdot g^{-1})
\end{aligned}$$

□

Claim 2. $\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1})$ is the well-defined (1,1)-form on S.

Proof. We take local coordinates (U, z_i) and (W, w_j) on S such that $U \cap W \neq \emptyset$. Let Jacobi matrix $J = (\frac{\partial w_i}{\partial z_j})$. We can let

$$(\omega_1 + \sqrt{-1}\omega_2) \mid_U = \overline{\partial}(\varphi_1 dz_1 + \varphi_2 dz_2) = \overline{\partial}\varphi_1 \wedge dz_1 + \bar{\partial}\varphi_2 \wedge dz_2$$

$$(\omega_1+\sqrt{-1}\omega_2)\mid_W=\overline{\partial}(\gamma_1 dw_1+\gamma_2 dw_2)=\overline{\partial}\gamma_1\wedge dw_1+\bar{\partial}\gamma_2\wedge dw_2$$

Then on $U\cap W$,

$$\begin{pmatrix} \bar{\partial}\gamma_1 & \bar{\partial}\gamma_2 \end{pmatrix}\wedge\begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} = \begin{pmatrix} \bar{\partial}\varphi_1 & \bar{\partial}\varphi_2 \end{pmatrix}\wedge\begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix}$$

So

$$\begin{pmatrix} \bar{\partial}\varphi_1 & \bar{\partial}\varphi_2 \end{pmatrix}=\begin{pmatrix} \bar{\partial}\gamma_1 & \bar{\partial}\gamma_2 \end{pmatrix}J \tag{3.6}$$

On the other hand, we have

$$g(z)=J^t g(w)\overline{J} \tag{3.7}$$

where $g(z)=(g_{i\bar{j}}(z))$ and $g(w)=(g_{i\bar{j}}(w))$ denote the metrics on U and W respectively. Then on $U\cap W$, from (3.6),(3.7), we calculate

$$\begin{aligned}
& \mathrm{tr}\begin{pmatrix} \bar{\partial}\gamma_1 \\ \bar{\partial}\gamma_2 \end{pmatrix}\wedge\begin{pmatrix} \partial\bar{\gamma}_1 & \partial\bar{\gamma}_2 \end{pmatrix}\cdot g^{-1}(w) \\
= & \ \mathrm{tr}\begin{pmatrix} \bar{\partial}\gamma_1 \\ \bar{\partial}\gamma_2 \end{pmatrix}\wedge\begin{pmatrix} \overline{\bar{\partial}\gamma_1} & \overline{\bar{\partial}\gamma_2} \end{pmatrix}\cdot g^{-1}(w) \\
= & \ \mathrm{tr}(J^t)^{-1}\begin{pmatrix} \bar{\partial}\varphi_1 \\ \bar{\partial}\varphi_2 \end{pmatrix}\wedge\begin{pmatrix} \overline{\bar{\partial}\varphi_1} & \overline{\bar{\partial}f_2} \end{pmatrix}\bar{J}^{-1}\cdot\bar{J}\cdot g^{-1}(z)\cdot J^t \\
= & \ \mathrm{tr}J^t\cdot(J^t)^{-1}\begin{pmatrix} \bar{\partial}\varphi_1 \\ \bar{\partial}\varphi_2 \end{pmatrix}\wedge\begin{pmatrix} \overline{\bar{\partial}\varphi_1} & \overline{\bar{\partial}\varphi_2} \end{pmatrix}\cdot g^{-1}(z) \\
= & \ \mathrm{tr}\begin{pmatrix} \bar{\partial}\varphi_1 \\ \bar{\partial}\varphi_2 \end{pmatrix}\wedge\begin{pmatrix} \partial\bar{\varphi}_1 & \partial\bar{\varphi}_2 \end{pmatrix}\cdot g^{-1}(z)
\end{aligned}$$

which proves that $\mathrm{tr}(\bar{\partial}A\wedge\partial A^*\cdot g^{-1})$ is the well-defined (1,1) form on S. □

□

Although $\mathrm{tr}\bar{\partial}A\wedge\partial A^*\cdot g^{-1}$ is the well-defined (1,1) form on S, we can not express it by ω_1 and ω_2. But in the particular case, we can do it.

Proposition 9. *When $\omega_2=n\omega_1$, $n\in\mathbb{Z}$,*

$$\mathrm{tr}\bar{\partial}A\wedge\partial A^*\cdot g^{-1}=\frac{\sqrt{-1}}{4}(1+n^2)\parallel\omega_1\parallel^2_{\omega_S}\omega_S \tag{3.8}$$

where ω_S is the given Calabi-Yau metric on S.

Proof. We recall that locally, we have

$$\begin{aligned}
\omega_1 &= \bar{\partial}\xi, \quad \xi=\xi_1 dz_1+\xi_2 dz_2, \\
\omega_2 &= \bar{\partial}\zeta, \quad \zeta=\zeta_1 dz_1+\zeta_2 dz_2, \\
\varphi_j &= \xi_j+\sqrt{-1}\zeta_j, \quad \text{for } \ j=1,2 \\
A &= \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \ A^*=\begin{pmatrix} \bar{\varphi}_1 & \bar{\varphi}_2 \end{pmatrix}.
\end{aligned}$$

When $\omega_2=n\omega_1$,we take $\zeta=n\xi$. Then $\bar{\partial}\zeta_j=n\bar{\partial}\xi_j$. So

$$\bar{\partial}A=\begin{pmatrix} \bar{\partial}\varphi_1 \\ \bar{\partial}\varphi_2 \end{pmatrix}=(1+n\sqrt{-1})\begin{pmatrix} \bar{\partial}\xi_1 \\ \bar{\partial}\xi_2 \end{pmatrix}$$

and

$$\partial A^*=\begin{pmatrix} \partial\bar{\varphi}_1 & \partial\bar{\varphi}_2 \end{pmatrix}=(1-n\sqrt{-1})\begin{pmatrix} \partial\bar{\xi}_1 & \partial\bar{\xi}_2 \end{pmatrix}$$

Using above equalities, locally we calculate

$$
\begin{aligned}
&\mathrm{tr}\bar{\partial}A\wedge\partial A^*\cdot g^{-1}\\
=&(1+n^2)\mathrm{tr}\begin{pmatrix}\bar{\partial}\xi_1\\ \bar{\partial}\xi_2\end{pmatrix}\wedge\begin{pmatrix}\partial\bar{\xi}_1 & \partial\bar{\xi}_2\end{pmatrix}\cdot g^{-1}\\
=&\frac{1+n^2}{\det g}\mathrm{tr}\begin{pmatrix}\frac{\partial\xi_1}{\partial\bar{z}_i}d\bar{z}_i\\ \frac{\partial\xi_2}{\partial\bar{z}_i}d\bar{z}_i\end{pmatrix}\wedge\begin{pmatrix}\overline{\frac{\partial\xi_1}{\partial\bar{z}_j}}dz_j & \overline{\frac{\partial\xi_2}{\partial\bar{z}_j}}dz_j\end{pmatrix}\cdot\begin{pmatrix}g_{2\bar{2}} & -g_{1\bar{2}}\\ -g_{2\bar{1}} & g_{1\bar{1}}\end{pmatrix}\\
=&\frac{1+n^2}{\det g}\mathrm{tr}\begin{pmatrix}\frac{\partial\xi_1}{\partial\bar{z}_i}\\ \frac{\partial\xi_2}{\partial\bar{z}_i}\end{pmatrix}\wedge\begin{pmatrix}\overline{\frac{\partial\xi_1}{\partial\bar{z}_j}} & \overline{\frac{\partial\xi_2}{\partial\bar{z}_j}}\end{pmatrix}\cdot\begin{pmatrix}g_{2\bar{2}} & -g_{1\bar{2}}\\ -g_{2\bar{1}} & g_{1\bar{1}}\end{pmatrix}d\bar{z}_i\wedge dz_j
\end{aligned}
\tag{3.9}
$$

In order to get the global formula, we need some formulas about ω_1. Because ω_1 is real, we have

$$
\overline{\frac{\partial\xi_i}{\partial\bar{z}_j}}=-\frac{\partial\xi_j}{\partial\bar{z}_i}\quad\text{for}\quad i,j=1,2
\tag{3.10}
$$

From ω_1 is anti-self-dual, i.e., $\omega_1\wedge\omega_S=0$, locally we have

$$
g_{1\bar{1}}\frac{\partial\xi_2}{\partial\bar{z}_2}+g_{2\bar{2}}\frac{\partial\xi_1}{\partial\bar{z}_1}-g_{1\bar{2}}\frac{\partial\xi_2}{\partial\bar{z}_1}-g_{2\bar{1}}\frac{\partial\xi_1}{\partial\bar{z}_2}=0
\tag{3.11}
$$

Because

$$
\omega_1\wedge\omega_1=-\omega_1\wedge *\omega_1=-\omega_1 * \bar{\omega}_1=-\parallel\omega_1\parallel^2\frac{\omega_S^2}{2!},
\tag{3.12}
$$

locally we also have

$$
\frac{1}{\det(g)}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\frac{\partial\xi_2}{\partial\bar{z}_2}-\frac{\partial\xi_1}{\partial\bar{z}_2}\frac{\partial\xi_2}{\partial\bar{z}_1}\right)=\frac{1}{8}\parallel\omega_1\parallel^2
\tag{3.13}
$$

Now using above (3.10), (3.11) and (3.13), we can calculate the component of $d\bar{z}_1\wedge dz_1$ in (3.9):

$$
\begin{aligned}
&\frac{1+n^2}{\det(g)}\left(g_{2\bar{2}}\frac{\partial\xi_1}{\partial\bar{z}_1}\overline{\frac{\partial\xi_1}{\partial\bar{z}_1}}-g_{2\bar{1}}\frac{\partial\xi_1}{\partial\bar{z}_1}\overline{\frac{\partial\xi_2}{\partial\bar{z}_1}}-g_{1\bar{2}}\frac{\partial\xi_2}{\partial\bar{z}_1}\overline{\frac{\partial\xi_1}{\partial\bar{z}_1}}-g_{1\bar{1}}\frac{\partial\xi_2}{\partial\bar{z}_1}\overline{\frac{\partial\xi_2}{\partial\bar{z}_1}}\right)\\
=&\frac{1+n^2}{\det(g)}\left(g_{2\bar{1}}\frac{\partial\xi_1}{\partial\bar{z}_1}\frac{\partial\xi_1}{\partial\bar{z}_2}+g_{1\bar{2}}\frac{\partial\xi_2}{\partial\bar{z}_1}\frac{\partial\xi_1}{\partial\bar{z}_1}-g_{2\bar{2}}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\right)^2-g_{1\bar{1}}\frac{\partial\xi_2}{\partial\bar{z}_1}\frac{\partial\xi_1}{\partial\bar{z}_2}\right)\\
=&\frac{1+n^2}{\det(g)}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\left(g_{2\bar{1}}\frac{\partial\xi_1}{\partial\bar{z}_2}+g_{1\bar{2}}\frac{\partial\xi_2}{\partial\bar{z}_1}\right)-g_{2\bar{2}}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\right)^2-g_{1\bar{1}}\frac{\partial\xi_2}{\partial\bar{z}_1}\frac{\partial\xi_1}{\partial\bar{z}_2}\right)\\
=&\frac{1+n^2}{\det(g)}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\left(g_{1\bar{1}}\frac{\partial\xi_2}{\partial\bar{z}_2}+g_{2\bar{2}}\frac{\partial\xi_1}{\partial\bar{z}_1}\right)-g_{2\bar{2}}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\right)^2-g_{1\bar{1}}\frac{\partial\xi_2}{\partial\bar{z}_1}\frac{\partial\xi_1}{\partial\bar{z}_2}\right)\\
=&\frac{1+n^2}{\det(g)}g_{1\bar{1}}\left(\frac{\partial\xi_1}{\partial\bar{z}_1}\frac{\partial\xi_2}{\partial\bar{z}_2}-\frac{\partial\xi_2}{\partial\bar{z}_1}\frac{\partial\xi_1}{\partial\bar{z}_2}\right)\\
=&\frac{1+n^2}{8}\parallel\omega_1\parallel^2 g_{1\bar{1}}
\end{aligned}
\tag{3.14}
$$

As the same reason, the components of $d\bar{z}_2 \wedge dz_1$, $d\bar{z}_1 \wedge dz_2$ and $d\bar{z}_2 \wedge dz_2$ in (3.9) are $\frac{1+n^2}{8} \parallel \omega_1 \parallel^2 g_{1\bar{2}}$, $\frac{1+n^2}{8} \parallel \omega_1 \parallel^2 g_{2\bar{1}}$ and $\frac{1+n^2}{8} \parallel \omega_1 \parallel^2 g_{2\bar{2}}$ respectively. So we obtain

$$\begin{aligned}
& \mathrm{tr}\bar{\partial} A \wedge \partial A^* \cdot g_S^{-1} \\
= & \frac{1+n^2}{8} \parallel \omega_1 \parallel^2 (g_{1\bar{1}} d\bar{z}_1 \wedge dz_1 + g_{1\bar{2}} d\bar{z}_2 \wedge dz_1 + g_{2\bar{1}} d\bar{z}_1 \wedge dz_2 + g_{2\bar{2}} d\bar{z}_2 \wedge dz_2) \\
= & \frac{\sqrt{-1}}{4}(1+n^2) \parallel \omega_1 \parallel^2 \omega_S.
\end{aligned}$$

□

4. Constructing the reducible solution to Strominger's system

We take the 3-dimensional hermitian manifolds $(X, \omega_0, \tilde{I})$ as described in section 2. Let ω_S is the Calabi-Yau metric on S. From (2.3) and (3.2), the Kahler form ω_0 is

$$\omega_0 = \pi^* \omega_S + \frac{\sqrt{-1}}{2} \theta \wedge \bar{\theta}.$$

Using (3.3) and the facts that $\parallel \Omega \parallel = 1$ and ω_1 and ω_2 are anti-self-dual, we have

$$\begin{aligned}
& d(\parallel \Omega \parallel_{\omega_0} \omega_0^2) \\
& = d\omega_0^2 = d(\pi^* \omega_S^2 + \sqrt{-1}\pi^* \omega_S \wedge \theta \wedge \bar{\theta}) \\
& = \sqrt{-1}\pi^* \omega_S \wedge d\theta \wedge \bar{\theta} - \sqrt{-1}\pi^* \omega_S \wedge \theta \wedge d\bar{\theta} \qquad (4.1) \\
& = \sqrt{-1}\pi^* \omega_S \wedge (\omega_1 + \sqrt{-1}\omega_2) \wedge \bar{\theta} - \sqrt{-1}\pi^* \omega_S \wedge (\omega_1 - \sqrt{-1}\omega_2) \wedge \theta \\
& = 0
\end{aligned}$$

According to Lemma 1, (ω_0, Ω) is the solution of equation (1.4). Let u be the smooth function on S and take

$$\omega_u = \pi^*(e^u \omega_S) + \frac{\sqrt{-1}}{2} \theta \wedge \bar{\theta} \qquad (4.2)$$

Then

$$\parallel \Omega \parallel_{\omega_u}^2 = \frac{\omega_0^3}{\omega_u^3} = \frac{1}{e^{2u}}$$

and

$$\begin{aligned}
\parallel \Omega \parallel_{\omega_u} \omega_u^2 &= e^{-u}(e^{2u}\omega_S^2 + \sqrt{-1} e^u \omega_S \wedge \theta \wedge \bar{\theta}) \\
&= \omega_0^2 + (e^u - 1)\omega_S^2
\end{aligned}$$

From (4.1), we obtain

$$d(\parallel \Omega \parallel_{\omega_u} \omega_u^2) = d\omega_0^2 + d(e^u - 1) \wedge \omega_S^2 = 0$$

and we have proven the following

Lemma 10. [9] *The metric (4.2) defined over X satisfies the equation (1.5) and so satisfies the equation (1.4).*

Now we construct the solutions to Strominger's system. Because $\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi} \in H^{1,1}(S, \mathbb{R}) \cap H^2(S, \mathbb{Z})$, there are holomorphic line bundles L_1 and L_2 over S such that their curvatures of hermitian connections are $\sqrt{-1}\omega_1$ and $\sqrt{-1}\omega_2$ respectively. Corresponding to these curvatures, there exist hermitian metrics h_1 and h_2 on L_1 and L_2 because S is Kahler. Let

$$E = L_1 \oplus L_2 \oplus T'S \qquad (4.3)$$

and let

$$H_0 = (h_1, h_2, \omega_S).$$

The curvature F_{H_0} of E is

$$F_{H_0} = \begin{pmatrix} \sqrt{-1}\omega_1 & & \\ & \sqrt{-1}\omega_2 & \\ & & R_S \end{pmatrix} \tag{4.4}$$

where R_S is the curvature of $T'S$ corresponding to the hermitian metric ω_S. Let $V = \pi^* E$, and let $F_{\tilde{H}_0} = \pi^* F_{H_0}$. We try to make $(V, F_{\tilde{H}_0}, X, \omega_u)$ to be the solution to Strominger's system.

Lemma 11. *For any smooth function u on S, $(V, F_{\tilde{H}_0}, X, \omega_u)$ satisfies the equations (1.1), (1.2) and (1.4).*

Proof. By lemma 10, equation (1.4) is satisfied. Because ω_1, ω_2 and R_S are (1,1) forms on S, $F_{\tilde{H}_0}$ is the (1,1) form on X. So $F^{2,0}_{\tilde{H}_0} = F^{0,2}_{\tilde{H}_0} = 0$. We also have

$$F_{\tilde{H}_0} \wedge \omega_u^2 = \pi^*(F_{H_0} \wedge e^u \omega_S) \wedge (\pi^*(e^u\omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}) = 0$$

by facts $\omega_1 \wedge \omega_S = \omega_2 \wedge \omega_S = 0$ and $R_S \wedge \omega_S = 0$. □

So we only need to consider the equation (1.3). We take the factor $\alpha' = \frac{1}{2}$[3], then $(V, F_{\tilde{H}_0}, X, \omega_u)$ should satisfy the equation

$$\sqrt{-1}\partial\bar{\partial}\omega_u = \frac{1}{2}(\mathrm{tr} F_{\tilde{H}_0} \wedge F_{\tilde{H}_0} - \mathrm{tr} R_u \wedge R_u) \tag{4.5}$$

here R_u denotes the curvature of Hermitian connection on $T'X$ corresponding to the hermitian metric ω_u. Now we calculate each term in equation (4.5). The Laplace operator $\triangle$ on S is define by $\triangle = \bar{\partial}^* \circ \bar{\partial}$ associate the Calabi-Yau metric ω_S. So for any smooth function ψ on S,

$$\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega_S = \triangle\psi \cdot \frac{\omega_S^2}{2!} \tag{4.6}$$

Claim 3. $\sqrt{-1}\partial\bar{\partial}\omega_u = \triangle e^u \cdot \frac{\omega_S^2}{2!} + \frac{1}{2}(\| \omega_1 \|^2 + \| \omega_2 \|^2)\frac{\omega_S^2}{2!}$

Proof. Using (3.3), (4.6) and (3.12), we get

$$\begin{aligned}
\sqrt{-1}\partial\bar{\partial}\omega_u &= \sqrt{-1}\partial\bar{\partial}(e^u\omega_S + \frac{\sqrt{-1}}{2}\theta \wedge \bar{\theta}) \\
&= \sqrt{-1}\partial\bar{\partial}e^u \wedge \omega_S - \frac{1}{2}\bar{\partial}\theta \wedge \partial\bar{\theta} \\
&= \triangle e^u \cdot \frac{\omega_S^2}{2!} - \frac{1}{2}(\omega_1 + \sqrt{-1}\omega_2) \wedge (\omega_1 - \sqrt{-1}\omega_2) \\
&= \triangle e^u \cdot \frac{\omega_S^2}{2!} - \frac{1}{2}(\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2) \\
&= \triangle e^u \cdot \frac{\omega_S^2}{2!} + \frac{1}{2}(\| \omega_1 \|^2 + \| \omega_2 \|^2)\frac{\omega_S^2}{2!}
\end{aligned}$$

□

Clam 4. $\mathrm{tr} F_{H_0} \wedge F_{H_0} = (\| \omega_1 \|^2 + \| \omega_2 \|^2)\frac{\omega_S^2}{2!} + \mathrm{tr} R_S \wedge R_S$

[3]We can take $\alpha' = 1$, only if we take the metric $2\omega_u$.

Proof. From (4.4), we have

$$\begin{aligned}\mathrm{tr}F_{H_0}\wedge F_{H_0} &= -\omega_1\wedge\omega_1-\omega_2\wedge\omega_2+\mathrm{tr}R_S\wedge R_S\\ &= (\|\omega_1\|^2+\|\omega_2\|^2)\frac{\omega_S^2}{2!}+\mathrm{tr}R_S\wedge R_S\end{aligned}$$

□

Claim 5. $\mathrm{tr}R_u\wedge R_u=\pi^*\mathrm{tr}R_S\wedge R_S+2\pi^*(\partial\bar\partial u\wedge\partial\bar\partial u)+2\pi^*(\partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g_S))\wedge\omega)$.

Proof. Actually in the proof of the Proposition 8 we don't use the condition that ω_S is Kahler. So if we replace metric g by $e^u g$, we can still get:

$$\begin{aligned}\mathrm{tr}R_u\wedge R_u =&\pi^*(\mathrm{tr}R_S^u\wedge R_S^u+2\mathrm{tr}\partial\bar\partial(\bar\partial A\wedge\partial A^*\cdot(e^ug)^{-1}))\\ =&\pi^*(\mathrm{tr}R_S^u\wedge R_S^u+2\partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g^{-1})))\end{aligned}\tag{4.7}$$

here R_S^u denotes the curvature of hermitian connection of $T'S$ corresponding to the hermitian metric e^ug. So

$$\begin{aligned}R_S^u &= \bar\partial(\partial(e^ug)\cdot(e^ug)^{-1})\\ &= \bar\partial(\partial u\cdot I+\partial g\cdot g^{-1})\\ &= \bar\partial\partial u\cdot I+R_S\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}R_S^u\wedge R_S^u =&\mathrm{tr}R_S\wedge R_S+2\partial\bar\partial u\wedge\partial\bar\partial u+2\partial\bar\partial u\wedge\mathrm{tr}R_S\\ =&\mathrm{tr}R_S\wedge R_S+2\partial\bar\partial u\wedge\partial\bar\partial u\end{aligned}\tag{4.8}$$

here we use the fact that $\mathrm{tr}R_S=0$ because the hermitian metric g is the Calabi-Yau metric on S. Inserting (4.8) into (4.7), we have proven the claim. □

From Claim 3, Claim 4, and Claim 5, we can rewrite the equation (4.5) as

$$\triangle e^u\cdot\frac{\omega_S^2}{2!}+\partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g^{-1}))+\partial\bar\partial u\wedge\partial\bar\partial u=0\tag{4.9}$$

Now we can prove

Theorem 12. *$(V,F_{\pi^*H_0},X,\omega_u)$ is the solution of Strominger's system if and only if the function u of S satisfies the equation (4.9). In particular, when $\omega_2=n\omega_1$, $n\in\mathbb{Z}$, $(V,F_{\pi^*H_0},X,\omega_u)$ is the solution to Strominger's system if and only if smooth function u on S satisfies the equation:*

$$\triangle\left(e^u+\frac{(1+n^2)}{4}\|\omega_1\|^2_{\omega_S}e^{-u}\right)\frac{\omega_S^2}{2!}+\partial\bar\partial u\wedge\partial\bar\partial u=0\tag{4.10}$$

or

$$\triangle\left(e^u+\frac{(1+n^2)}{4}\|\omega_1\|^2_{\omega_S}e^{-u}\right)-8\frac{\det(u_{i\bar j})}{\det(g_{i\bar j})}=0\tag{4.11}$$

Proof. We only need to prove the second part of theorem. When $\omega_2=n\omega_1$, from Proposition 9 and (4.6), we have

$$\begin{aligned}\partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g^{-1})) &= \frac{\sqrt{-1}}{4}(1+n^2)\partial\bar\partial(e^{-u}\|\omega_1\|^2)\wedge\omega_S\\ &= \frac{(1+n^2)}{4}\triangle(e^{-u}\|\omega_1\|^2)\frac{\omega_S^2}{2!}\end{aligned}$$

So the equation (4.10) follows from equation (4.9). Equation (4.11) is derived from

$$\begin{aligned}\partial\bar{\partial}u\wedge\partial\bar{\partial}u &= 2\det(u_{i\bar{j}})dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\\ &= -8\frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})}\frac{\omega_S^2}{2!}.\end{aligned}$$

□

5. IRREDUCIBLE SOLUTION

We have constructed the following reducible solution: (V,π^*H_0,X,ω_u) if $u\in C^\infty(S,\mathbb{R})$ is the solution of the equation

$$\sqrt{-1}\partial\bar{\partial}e^u\wedge\omega_S+\partial\bar{\partial}(e^{-u}\mathrm{tr}(\bar{\partial}A\wedge\partial A^*\cdot g^{-1}))+\partial\bar{\partial}u\wedge\partial\bar{\partial}u=0 \tag{5.1}$$

satisfying elliptic condition

$$e^u\omega_S+\sqrt{-1}e^{-u}\mathrm{tr}(\bar{\partial}A\wedge\partial A^*\cdot g^{-1})-2\sqrt{-1}\partial\bar{\partial}u>0 \tag{5.2}$$

and normalization

$$\int_S e^{-u}=A.$$

From the next section to the end of the paper, we will prove that the equation (5.1) has a unique solution if $A<<1$. So in this section, we assume that u is the solution of equation (5.1). We will obtain the irreducible solution by perturbing around the reducible solution (V,π^*H_0,X,ω_u). We follow the Li-Yau's method [15].

Let D''_s be a family of holomorphic structures on E over S, H be a hermitian metric on E over S and ω be a hermitian metric on S. We want to look for conditions on D''_s, H and and function ϕ such that under these conditions $(V,\pi^*D''_s,\pi^*H,X,\omega_{u+\phi})$ is the solution to Strominger's solution, where

$$\omega_{u+\phi}=\pi^*(e^{u+\phi}\omega)+\frac{\sqrt{-1}}{2}\theta\wedge\bar{\theta}.$$

Fix the metric H_0 as the reference metric on E over S. Then for any hermitian metric H on E, we can define a smooth endomorphism h on E by

$$<s,t>_H=<s\cdot h,t>_{H_0}.$$

Under this isomorphism, we define $\mathcal{H}(E)_1$ be the space of all hermitian metric on E whose corresponding endomorphism has determinant one. Let $\mathcal{C}(\omega_S)=\{e^\phi\omega_S\}$ be the space of all hermitian metrics on S which are conformal to ω_S. Let $\mathrm{End}^0 E$ be the vector bundle of traceless hermitian anti-symmetric endomorphisms of (E,H_0). Let $\mathcal{H}_0(S)=\left\{\psi\frac{\omega^2}{2!}\mid\int_S\psi\frac{\omega^2}{2!}=0\right\}$. We define the operator

$$\mathbf{L}=\mathbf{L_1}\oplus\mathbf{L_2}:\mathcal{H}_1(E)\times\mathcal{C}(\omega_S)\to\Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)\oplus\mathcal{H}_0(S)$$

by

$$\mathbf{L_1}(h,e^\phi\omega_S)=e^\phi h^{-\frac{1}{2}}F_hh^{\frac{1}{2}}\wedge\omega_S \tag{5.3}$$

$$\begin{aligned}\mathbf{L_2}(h,e^\phi\omega_S)=&\sqrt{-1}\partial\bar{\partial}(e^{u+\phi}\omega_S)+\partial\bar{\partial}(e^{-u-\phi}\mathrm{tr}(\bar{\partial}A\wedge\partial A^*\cdot g^{-1}_{\omega_S}))\\ &+\partial\bar{\partial}(u+\phi)\wedge\partial\bar{\partial}(u+\phi)-\frac{1}{2}(\mathrm{tr}F_h\wedge F_h-\mathrm{tr}F_{H_0}\wedge F_{H_0})\end{aligned} \tag{5.4}$$

Because $F_h = F_{H_0} + D_0''(D_0'h \cdot h^{-1})$ and $F_{H_0} \wedge \omega_S = 0$, then

$$e^{\phi}h^{-1/2}F_h h^{1/2} \wedge \omega_S = e^{\phi}D_0''(D_0'h \cdot h^{-1}) \wedge \omega_S$$

and from the notation of paper [20], we get

$$\mathrm{tr}(e^{\phi}h^{-1/2}F_h h^{1/2} \wedge \omega_S) = e^{\phi}D_0''(D_0'\mathrm{tr}\log h) \wedge \omega_S = 0$$

because $\det h = 1$. So the image of $\mathbf{L_1}$ lies in $\Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)$. As to $\mathbf{L_2}$, according to $\partial\bar{\partial}$-lemma on K3 surface, the image of $\mathbf{L_2}$ lies in $R(dd_c)$. Because for any $\sqrt{-1}\partial\bar{\partial}\alpha \in R(\sqrt{-1}\partial\bar{\partial})$, we can write $\sqrt{-1}\partial\bar{\partial}\alpha = \psi\frac{\omega_S^2}{2!}$ for some function ψ such that $\int_S \psi\frac{\omega^2}{2!} = 0$. So the image of $\mathbf{L_2}$ lies in $\mathcal{H}_0(S)$. Therefore the operator $\mathbf{L}$ is well-defined.

Proposition 13. *If $(h, e^{\phi}\omega_S) \in \ker \mathbf{L}$, then $(V, \pi^*D_0'', \pi^*h, \omega_{u+\phi})$ is the solution of Strominger's system.*

Proof. According to paper [15], $(V, \pi^*D_0'', \pi^*h, X, \omega_{u+\phi})$ is the solution to Strominger's system if and only if $(\pi^*h, \omega_{u+\phi})$ lies in the kernel of the operator:

$$\tilde{\mathbf{L}} = \tilde{\mathbf{L}}_\mathbf{1} \oplus \tilde{\mathbf{L}}_\mathbf{2} \oplus \tilde{\mathbf{L}}_\mathbf{3} : \mathcal{H}(V)_1 \times \mathcal{H}(X) \to \Omega^6_{\mathbb{R}}(\mathrm{End}^0 V) \oplus R(dd_c) \oplus R(d^*_{\omega_0})$$

defined by

$$\begin{aligned}
\tilde{\mathbf{L}}_\mathbf{1}(\tilde{h}, \tilde{\omega}) &= \tilde{h}^{-\frac{1}{2}}F_{\tilde{h}}\tilde{h}^{\frac{1}{2}} \wedge \tilde{\omega}^2 \\
\tilde{\mathbf{L}}_\mathbf{2}(\tilde{h}, \tilde{\omega}) &= \sqrt{-1}\partial\bar{\partial}\tilde{\omega} - \frac{1}{2}(\mathrm{tr}F_{\tilde{h}} \wedge F_{\tilde{h}} - \mathrm{tr}R_{\tilde{\omega}} \wedge R_{\tilde{\omega}}) \\
\tilde{\mathbf{L}}_\mathbf{3}(\tilde{h}, \tilde{\omega}) &= *_{\omega_0} d(\parallel \Omega \parallel_{\tilde{\omega}} \tilde{\omega}^2)
\end{aligned}$$

We want to reduce above operator $\tilde{\mathbf{L}}$ to vector bundle E over S. When $(\tilde{h}, \tilde{\omega}) = (\pi^*h, \omega_{u+\phi})$,

$$\begin{aligned}
\tilde{\mathbf{L}}_\mathbf{1}(\pi^*h, \omega_{u+\phi}) &= \pi^*(h^{-\frac{1}{2}}F_h h^{\frac{1}{2}}) \wedge (\omega_{u+\phi})^2 \\
&= \pi^*(h^{-\frac{1}{2}}F_h h^{\frac{1}{2}} \wedge (e^{u+\phi}\omega_S)) \wedge (\pi^*(e^{u+\phi}\omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}) \\
&= \pi^*e^u \cdot \pi^*\mathbf{L_1}(h, e^{\phi}\omega_S) \wedge (\pi^*(e^{u+\phi}\omega_S) + \sqrt{-1}\theta \wedge \bar{\theta}),
\end{aligned}$$

then $(h, e^{\phi}\omega_S) \in \ker \mathbf{L_1}$ if and only if $(\pi^*h, \omega_{u+\phi}) \in \ker \tilde{\mathbf{L}}_\mathbf{1}$.

Next we consider $\tilde{\mathbf{L}}_\mathbf{2}$. When $(\tilde{h}, \tilde{\omega}) = (\pi^*h, \omega_{u+\phi})$, by Proposition 8 and as explained in section 4, we have

$$\mathrm{tr}R_{\omega_{u+\phi}} \wedge R_{\omega_{u+\phi}} = \pi^*(\mathrm{tr}R_{e^{(u+\phi)}\omega_S} \wedge R_{e^{(u+\phi)}\omega_S} + 2\partial\bar{\partial}(e^{-u-\phi}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1}_{\omega_S})))$$

Meanwhile we have

$$R_{e^{(u+\phi)}\omega_S} = \bar{\partial}\partial(u+\phi) \cdot I + R_{\omega_S}$$

and thus

$$\mathrm{tr}R_{e^{(u+\phi)}\omega_S} \wedge R_{e^{(u+\phi)}\omega_S} = 2\partial\bar{\partial}(u+\phi) \wedge \partial\bar{\partial}(u+\phi) + \mathrm{tr}R_{\omega_S} \wedge R_{\omega_S}$$

because $\mathrm{tr}R_{\omega_S} = 0$. Then we see that

$$\begin{aligned}
\mathrm{tr}R_{\omega_{u+\phi}} \wedge R_{\omega_{u+\phi}} &= \pi^*\{2\partial\bar{\partial}(u+\phi) \wedge \partial\bar{\partial}(u+\phi) + \mathrm{tr}(R_{\omega_S} \wedge R_{\omega_S}) \\
&\quad +2\partial\bar{\partial}(e^{-u-\phi}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1}_{\omega_S}))\}
\end{aligned}$$

Using above equality, we can obtain

$$\begin{aligned}
\tilde{\mathbf{L}}_\mathbf{2}(\pi^*h, \omega^u) &= \pi^*\{\sqrt{-1}\partial\bar{\partial}(e^{(u+\phi)}\omega_S) + \partial\bar{\partial}(e^{-u-\phi}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1}_{\omega_S})) \\
&\quad +\partial\bar{\partial}(u+\phi) \wedge \partial\bar{\partial}(u+\phi) - \frac{1}{2}(\mathrm{tr}F_h \wedge F_h - \mathrm{tr}F_{H_0} \wedge F_{H_0})\} \\
&= \pi^*(\mathbf{L_2}(h, e^{\phi}\omega_S))
\end{aligned}$$

So $\tilde{\mathbf{L}}_{\mathbf{2}}(\pi^* h, e^u\omega) = 0$ if and only if $\mathbf{L}_{\mathbf{2}}(h,\omega) = 0$.

As to $\tilde{\mathbf{L}}_{\mathbf{3}}$, by Lemma 10, we always have $\tilde{\mathbf{L}}_{\mathbf{3}}(\pi^* h, \omega_{u+\phi}) = *_0 d(\| \Omega \|_{\omega_{u+\phi}} \omega_{u+\phi}^2) = 0$. $\square$

Certainly, the statement that $(V, \pi^* D_0, \pi^* H_0, \omega_u)$ is the solution is equivalent to say that $(I, \omega_S) \in \ker \mathbf{L}$. Now we follow Li and Yau's paper [15]. Let $\mathbb{R}^{+2} = \{T = (T_1, T_2) \in \mathbb{R}^2 \mid T_i > 0\}$; let I_1, I_2 and I_3 be the identity endomorphisms of L_1, L_2 and $T'S$ respectively. Then the assignment

$$T = (T_1, T_2) \in \mathbb{R}^{+2} \longmapsto h_T = T_1 I_1 \oplus T_2 I_2 \oplus T_1^{-1/2} T_2^{-1/2} I_3$$

associated each $T \in \mathbb{R}^{+2}$ to a hermitian endomorphism of E. Obviously, the hermitian curvature $F_{h_T} = F_{H_0}$. Therefore we still have $(h_T, \omega_S) \in \ker \mathbf{L}$.

Pick an integer k and a large p and endow the domain and the target of $\mathbf{L}_{\mathbf{1}} \oplus \mathbf{L}_{\mathbf{2}}$ the Banach space structures as indicated:

$$\mathcal{H}_1(E)_{L_k^p} \times \mathcal{C}(\omega_S)_{L_k^p} \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}} \oplus \mathcal{H}_0(S)_{L^p_{k-2}}$$

$\mathbf{L}_{\mathbf{1}} \oplus \mathbf{L}_{\mathbf{2}}$ becomes a smooth operator and its linearized operator $\delta\mathbf{L}_{\mathbf{1}} \oplus \delta\mathbf{L}_{\mathbf{2}}$ at a solution (h_T, ω_S) becomes a linear map

$$\Omega^0(\mathrm{Her}^0 E)_{L_k^p} \oplus \{\phi\omega_S\}_{L_k^p} \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}} \oplus \mathcal{H}_0(S)_{L^p_{k-2}}$$

Here we used $\mathrm{Her}^0 E$ to denote the $\mathbb{R}$-sub-vector bundle of $\mathrm{End}E$ consisting of traceless pointwise $<,>$-hermitian symmetric endormorphisms of E and the canonical isomorphisms $T_{h_T}\mathcal{H}_1(E)_{L_k^p} \cong \Omega^0(\mathrm{Her}^0 E)_{L_k^p}$. Clearly we also have $T_{\omega_S}\mathcal{C}(\omega_S)_{L_k^p} \cong \{\phi\omega_S\}_{L_k^p}$.

To study the kernel and the cokernel of $\delta\mathbf{L}_{\mathbf{1}} \oplus \delta\mathbf{L}_{\mathbf{2}}$ at a trivial solution (h_T, ω_S) we will first look at the linear map

$$F(\delta h) = D_0'' D_{0,h_T}'(\delta h) \wedge \omega_S : \Omega^0(\mathrm{Her}^0 E)_{L_k^p} \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}}.$$

Here according to our convention, $D_{h_T} = D'_{0,h_T} \oplus D''_0$ is the hermitian connection of (E, D_0'', h_T) for a $T = (T_1, T_2) \in \mathbb{R}^{+2}$. Since $(E, D_0'') = L_1 \oplus L_2 \oplus TS$ and $\deg L_i = 0$, the above is a linear elliptic operator of index 0 whose kernel is

$$V_0 = \{M_1 \cdot I_1 \oplus M_2 \cdot I_2 \oplus -(M_1/2 + M_2/2) I_3\}$$

and whose cokernel is

$$V_1 = \omega_S^2 \cdot V_0 \subset \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}}. \tag{5.5}$$

We let $\mathbf{P}$ be the obvious projection

$$\mathbf{P} : \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}} \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}}/V_1.$$

Proposition 14. *Let (S, ω_S), Ω_S, H_0 and $T = (T_1, T_2) \in \mathbb{R}^{+2}$ be as before. Then the linear operator*

$$\begin{aligned}&\mathbf{P} \circ \delta\mathbf{L}_{\mathbf{1}}(h_T, \omega_S) \oplus \delta\mathbf{L}_{\mathbf{2}}(h_T, \omega_S) :\\ &\Omega^0(Her^0 E)_{L_k^p} \oplus \{\phi\omega_S\}_{L_k^p} \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0\, E)_{L^p_{k-2}}/V_1 \oplus \mathcal{H}_0(S)_{L^p_{k-2}}\end{aligned}$$

is surjective.

Proof. Because $F_{h_T} = F_{H_0}$ and $\delta\omega = \phi\omega_S$ for some smooth function ϕ,

$$h_T{}^{-\frac{1}{2}} F_{h_T} h_T^{\frac{1}{2}} \wedge \delta\omega = 0.$$

Then

$$\delta\mathbf{L}_{\mathbf{1}}(h_T, \omega_S)(\delta h, \delta\omega) = D_0'' D_{h_T}' \delta h \wedge \omega_S$$

So

$$\mathbf{P}\circ\delta\mathbf{L_1}(h_T,\omega_S):\Omega^0(\mathrm{Her}E)_{L^p_k}\oplus\{\phi\omega_S\}_{L^p_k}\to\Omega^4_{\mathbb{R}}(\mathrm{End}^0E)_{L^p_{k-2}}/V_1$$

is surjective and

$$\ker\mathbf{P}\circ\delta\mathbf{L_1}(h_T,\omega_S)=V_0\oplus\{\phi\omega_S\}_{L^p_k} \tag{5.6}$$

On the other hand, when $\delta\omega=\phi\omega_S$,

$$\begin{aligned}\delta\mathbf{L_2}(h_T,\omega_S)(\delta h,\phi\omega_S) &= \sqrt{-1}\partial\bar\partial(e^u\phi\omega_S)-\partial\bar\partial(e^{-u}\phi\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g_S^{-1}))\\ &\quad+2\partial\bar\partial u\wedge\partial\bar\partial\phi-\mathrm{tr}\delta F_{h_T}(\delta h)\wedge F_{h_T}\end{aligned}$$

Because $\mathrm{tr}\delta F_{h_T}(\delta h)\wedge F_{h_T}\in\mathcal{H}_0(S)$, we only need to prove that $\delta\mathbf{L_2}(h_T,\omega_S):0\oplus\{\phi\omega_S\}_{L^p_k}\to\mathcal{H}_0(S)_{L^p_{k-2}}$ is surjective because we have (5.6). So we should solve the following equation:

$$\sqrt{-1}\partial\bar\partial(e^u\phi\omega_S)-\partial\bar\partial(e^{-u}\phi\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g_S^{-1}))+2\partial\bar\partial u\wedge\partial\bar\partial\phi=\psi\frac{\omega_S^2}{2!} \tag{5.7}$$

for any $\psi\in L^p_{k-2}$ such that $\int\psi=0$. If we define the linear operator L from L^p_k to L^p_{k-2} by

$$L(\phi)=\sqrt{-1}\partial\bar\partial(e^u\phi\omega_S)-\partial\bar\partial(e^{-u}\phi\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g_S^{-1}))+2\partial\bar\partial u\wedge\partial\bar\partial\phi$$

then L is the linearization of equation (5.2) and we will prove $\ker L^*=\mathbb{R}$ and $\dim\ker L=1$ in section 6. So $\mathcal{H}_0(S)\perp\ker L^*$. From the elliptic condition, the operator L is elliptic. So the equation (5.7) has solution $\phi\in L^p_k$ for any $\psi\in L^p_{k-2}$ such that $\int\psi=0$. Thus we have proven that $P\circ\delta L_1\oplus\delta L_2$ is surjective and it's kernel is to $V_0\oplus\ker L$. □

Now we deform the holomorphic structure D_0''. The following proposition is due to Jun Li.

Proposition 15. *There is a family D_s'' of deformations of holomorphic structures of E so that its k-th order for $k<m$ Kodaira-Spencer class κ all vanish while its m-th order Kodaira-Spencer class has non-vanishing summands in $H^1(L_i^\vee\otimes TS)$ and $H^1(TS^\vee\otimes L_i)$.*

Proof. (Given by Jun Li) Because $\deg L_i=0$ and TS is slope stable with respect to the Hodge class ω_S, $L_i^\vee\otimes TS$ has no global sections. Using the Serre duality, $H^2(L_i\otimes TS)=0$ as well. Thus to compute $H^1(L_i^\vee\otimes TS)$, we use Riemann-Roch for K3 surfaces

$$\chi(L_i^\vee\otimes TS)=\frac{1}{2}c_1(L_i^\vee\otimes TS)^2+2\chi(\mathcal{O}_S)-c_2(L_i^\vee\otimes TS)$$

Because $c_1(L_i^\vee\otimes TS)=-2c_1(L_i)$ and $c_2(L_i^\vee\otimes TS)=c_2(TS)+c_1(L_i)^2=20+c_1(L_i)^2$, we have

$$h^1(L_i^\vee\otimes TS)=16-c_1(L_i)^2\geq 16.$$

Here the last inequality follows from $c_1(L_i)\cdot\omega_S=0$ and the Hodge index theorem.

For the same reason, we have

$$h^1(TS^\vee\otimes L_i)\geq 16.$$

Now consider the vector bundle

$$E=L_1\oplus L_2\oplus TS.$$

The extension group $\mathrm{Ext}^1(E,E)$ has summands

$$\mathrm{Ext}^1(L_i,TS)\quad\text{and}\quad\mathrm{Ext}^1(TS,L_i), \tag{5.8}$$

which are $H^1(L_i^\vee\otimes TS)$ and $H^1(TS^\vee\otimes L_i)$, respectively; hence are positive dimension. Thus we can find a direction $\eta\in\mathrm{Ext}^1(E,E)$ that has non-trivial components in the desired factors (5.8).

It remains to show that η can be realized as the tangent of an actual deformation. But this may not be true since the obstruction to deformation of E is the traceless part of $\mathrm{Ext}^2(E,E)$, which is isomorphic two copies of $H^2(\mathcal{O}_S) \cong \mathbb{C}$. What is true is that there is a family of deformations of holomorphic structures of E, denoted by $\bar{\partial}_s$ so that

$$\frac{d^k}{ds^k}\bar{\partial}_s \mid_{s=0}= 0, \quad k < m$$

and

$$\frac{d^m}{ds^m}\bar{\partial}_s \mid_{s=0}\neq 0$$

has non-trivial exponents in (5.8). □

Actually, we can define the Kuranish map $K : \mathcal{U} \to \mathrm{Ext}^2(E,E)$, where $\mathcal{U}$ is some open neighborhood of origin in $\mathrm{Ext}^1(E,E)$. K is the holomorphic map and the complex analytic variety $\mathcal{X} = K^{-1}(0)$ is the parametric space of all holomorphic structures on E near D_0''. Considering the dimensions of $\mathrm{Ext}^1(E,E)$ and $\mathrm{Ext}^2(E,E)$, we can choose an element

$$\eta = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \in \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$$

such that $C_{i3} \neq 0$ and $C_{3j} \neq 0$ for $1 \leq i,j \leq 2$ and η belongs to the tangent cone of $\mathcal{X}$ at the point D_0''. So there is a curve D_s'' of degree m of smooth deformation of the holomorphic structure D_0''. If we write

$$D_s'' = D_0'' + A_s, \qquad A_s \in \Omega^{0,1}(\mathrm{End}^0 E),$$

then $A_0^{(k)} = 0$ for $k < m$ and $A_0^{(m)} = \eta$. We assume that C_{ij} are D_0''-harmonic. Because $\mathrm{Pic}S$ is discrete, we can assume further that $\mathrm{tr}A_s = 0$ for all s.

With the connection forms A_s, the metric H_0 and Kahler form ω_S so chosen, we can now define operators

$$\mathbf{L_{s,1}} \oplus \mathbf{L_{s,2}} : \mathcal{H}_1(E)_{L_k^p} \times \mathcal{C}(\omega_S)_{L_k^p} \to \Omega^4_{\mathbb{R}}(\mathrm{End}^0\, E)_{L^p_{k-2}} \oplus \mathcal{H}_0(S)_{L^p_{k-2}}$$

with $\mathbf{L_{s,i}}$ defined as in (5.3) and (5.4) of which the curvature form F_h is replaced by the hermitian curvature of (E, D_s'', h):

$$F_{s,h} = D_{s,h} \circ D_{s,h}.$$

From paper [15], we have

$$F_{s,h} = F_s + D_s''(D_s'h \cdot h)$$

and

$$F_s = F_0 + (D_0'' + D_0')(A_s - A_s^*) - (A_s - A_s^*) \wedge (A_s - A_S^*)$$

Because $\mathrm{tr}A_s = 0$, $\det h = 1$ and $F_0 \wedge \omega_S = 0$,

$$\mathrm{tr}\mathbf{L_{s,1}}(h, e^\phi \omega_S) = e^\phi \mathrm{tr} h^{-1/2} F_{s,h} h^{1/2} \wedge \omega_S = 0$$

So $\mathbf{L_{s,1}}$ still lies in $\Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}}$. Let $\mathbf{P}$ be the projection from $\Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}} \oplus \mathcal{H}_0(S)_{L^p_{k-2}}$ to $\Omega^4_{\mathbb{R}}(\mathrm{End}^0 E)_{L^p_{k-2}}/V_1 \oplus \mathcal{H}_0(S)_{L^p_{k-2}}$. Because we have proven that the linearized operator of $\mathbf{P} \circ \mathbf{L_{0,1}} \oplus \mathbf{L_{0,2}}$ is surjective at (h_{T_0}, ω_S). Hence by the implicit theorem, for sufficiently small s there are smooth solutions $(h_{s,T}, \omega_{s,T})$ to $\mathbf{P} \circ \mathbf{L_{s,1}} \oplus \mathbf{L_{s,2}} = 0$ near (h_{T_0}, ω_S). We can assume that the solutions $(h_{s,T}, \omega_{s,T})$ can be parameterized by

$$(s,T) \in [0,a) \times B_\epsilon(T_{0,1}) \times B_\epsilon(T_{0,2})$$

where $T_0 = (T_{0,1}, T_{0,2})$. For simplicity, we denote by $F_{s,T}$ the curvature of the hermitian vector bundle $(E, D''_s, h_{s,T})$. By our construction, it satisfies

$$\mathbf{L_{s,1}}(H_{s,T}, \omega_{s,T}) \equiv 0 \mod V_1, \quad \mathbf{L_{s,2}}(H_{s,T}, \omega_{s,T}) = 0.$$

Hence to find solutions to $\mathbf{L_{s,1}} \oplus \mathbf{L_{s,2}} = 0$ it suffices to investigate the vanishing loci of the functional $\mathbf{r_i}(\mathbf{s}, \cdot)$ from $B_\epsilon(T_{0,i})$ to the Lie algebra $u(L_i)$ defined by

$$\mathbf{r}_i(s,T) = \int_X \big[\mathbf{L}_{s,T}(H_{s,T}, \omega_{s,T})\big]_i \wedge \omega_S \tag{5.9}$$

where $[\cdot]_i$ is the projection from $\Omega^\bullet_{\mathbb{R}}(\operatorname{End}^0 E)$ to $\Omega^\bullet_{\mathbb{R}}(u(L_i))$ and $u(L_i)$ is the bundle of $<,>$-hermitian antisymmetric endomorphisms of L_i.

We shall compute $\mathbf{r}_i^{(k)}(0,T)$ for all T and for all $k \le 2m$. Because $\omega_{s,T} \in \mathcal{C}(\omega_S)$, we can write $\omega_{s,T} = \phi_{s,T}\omega_S$ for some positive functions $\phi_{s,T}$ on S such that $\phi_{0,T} = 1$. Then we have

$$\frac{d^k}{ds^k}\,|_{s=0}\ \omega_{s,T} = \phi^{(k)}_{0,T}\omega_S. \tag{5.10}$$

On the other hand, since $(h_{s,T}, \omega_{s,T})$ are solutions to $\mathbf{L_{s,1}} \oplus \mathbf{L_{s,2}} = 0 \mod V_1$, there is a function $\mathbf{c}(s,T)$ taking values in V_1 with $\mathbf{c}(0,T) = 0$ so that

$$F_{s,T} \wedge \omega_{s,T} = h_{s,T}^{-\frac{1}{2}}\mathbf{c}(s,T)h_{s,T}^{\frac{1}{2}}. \tag{5.11}$$

We can write

$$\mathbf{c}(s,T) = \operatorname{diag}\left(M_1(s,T), M_2(s,T), -(M_1(s,T)/2 + M_2(s,T)/2)I_3\right) \cdot \omega_S^2$$

where $M_i(s,T)$ is the function only depending on s and T.

At first, we compute $\mathbf{r}_i^{(k)}(0,T)$ for any T and $k \le m-1$. When $k \le m-1$, $A_0^{(k)} = 0$. Then

$$F^{(k)}_{0,T} = \sum_{l=0}^{k} C_k^l D''_0[D'_0 h^{(l)}_{0,T} \cdot (h^{-1}_{0,T})^{(k-l)}]. \tag{5.12}$$

Because $D'_0 h_T = 0$, $\dot F_{0,T} = D''_0[D'_0 \dot h_{0,T} \cdot h_T^{-1}]$ and

$$\frac{d}{ds}\,|_{s=0}\left(h_{s,T}^{-1/2} F_{s,T} h_{s,T}^{1/2}\right) = h_{0,T}^{-1/2}\dot F_{0,T} h_{0,T}^{1/2}.$$

We also have $F_{0,T} \wedge \omega_S = 0$. Combining these equalities with (5.10),

$$\dot{\mathbf{r}}_i(0,T) = \int_S T_i^{-1/2}[\dot F_{0,T}]_i T_i^{1/2} \wedge \omega_S + \int_S T^{-1/2}[F_{0,T}]_i T^{1/2} \wedge \dot\omega_S = 0$$

On the other hand, taking derivative of s at $s = 0$ to (5.11) and couple $\mathbf{c}(0,T) = 0$, we have

$$\dot F_{0,T} \wedge \omega_S = h_T^{1/2}\dot{\mathbf{c}}(0,T)h_T^{-1/2}$$

and then

$$[\dot F_{0,T}]_i \wedge \omega_S = [\dot{\mathbf{c}}(0,T)]_i = \dot M_i(0,T)\omega_S^2$$

From

$$\int_S \dot M_i(0,T)\omega_S^2 = \int_S [\dot F_{0,T}]_i \wedge \omega_S = 0$$

we get $\dot M_i(0,T) = 0$. So we get

$$\dot{\mathbf{c}}(0,T) = 0, \quad \text{and} \quad \dot F_{0,T} \wedge \omega_S = (D''_0 D'_0 \dot h_{0,T}) \cdot h^{-1}_{0,T} \wedge \omega_S = 0$$

Thus

$$\dot{h}_{0,T} \in V_0 \quad \text{and} \quad D_0'\dot{h}_{0,T} = 0.$$

In this way, we can prove that

$$\mathbf{r}_i^{(k)}(0,T) = 0, \quad \mathbf{c}^{(k)}(0,T) = 0, \quad F_{0,T}^{(k)} \wedge \omega_S = 0 \quad D_0' h_{0,T}^{(k)} = 0 \ \text{ for any } \ k \leq m-1. \tag{5.13}$$

When $k = m$, because of (5.13) and (5.10), we obtain

$$\frac{d^m}{ds^m} \mid_{s=0} (h_{s,T}^{-1/2} F_{s,T} h_{s,T}^{1/2} \wedge \omega_{s,T}) = h_{0,T}^{-1/2} F_{0,T}^{(m)} h_{0,T}^{1/2} \wedge \omega_S$$

and

$$F_{0,T}^{(m)} = (D_0'' + D_0')(A_0^{(m)} - A_0^{*(m)}) + D_0''[D_0' h_{0,T}^{(m)}] \cdot h_{0,T}^{-1}$$

Then we get

$$\mathbf{r}_i^{(m)}(0,T) = \int_S [F_{0,T}^{(m)}]_i \wedge \omega_S = 0$$

Because $F_{0,T}^{(k)} \wedge \omega_S = 0$, $\mathbf{c}^{(k)}(0,T) = 0$ for $k \leq m-1$, from (5.11), we get

$$F_{0,T}^{(m)} \wedge \omega_S = h_T^{1/2} \mathbf{c}^{(m)}(0,T) h_T^{-1/2}$$

So using the same method of case $k = 1$, we still have

$$\mathbf{c}^{(m)}(0,T) = F_{0,T}^{(m)} \wedge \omega_S = 0$$

When $k < m$, $A_0^{(k)} = 0$ and we have proven $D_0' h_{0,T}^{(k)} = 0$. By direct computation, we see

$$F_{0,T}^{(m+k)} = (D_0'' + D_0')(A_0^{(m+k)} - A_0^{*(m+k)}) + D_0'' \left(\frac{d^{m+k}}{ds^{m+k}} \mid_{s=0} (D_s' h_{s,T} \cdot h_{s,T}^{-1}) \right)$$

Then we still can get

$$\mathbf{r}_i^{(m+k)}(0,T) = 0, \quad \mathbf{c}^{(m,k)}(0,T) = 0 \quad F_{0,T}^{(m+k)} \wedge \omega_S = 0 \ \text{ for } \ k < m.$$

At last we compute $\mathbf{r}^{(2m)}(0,T)$. Directly computing, we get

$$\begin{aligned} F_{0,T}^{(2m)} \quad = \quad & (D_0'' + D_0')(A_0^{(2m)} - A_0^{*(2m)}) - C_{2m}^m (A_0^{(m)} - A_0^{*(m)}) \wedge (A_0^{(m)} - A_0^{*(m)}) \\ & - C_{2m}^m [A_0^{(m)}, d^m/ds^m \mid_{s=0} (D_s' h_{s,T} \cdot h_{s,T}^{-1})] \\ & + D_0''(d^{2m}/ds^{2m} \mid_{s=0} (D_s' h_{s,T} \cdot h_{s,T}^{-1}) \end{aligned}$$

and

$$[A_0^{(m)}, d^m/ds^m \mid_{s=0} (D_s' h_{s,T} \cdot h_{s,T}^{-1})] = [A_0^{(m)}, [A_0^{*(m)}, h_{0,T}] h_{0,T}^{-1}] + [A_0^{(m)}, D_0' h_{0,T}^{(m)} \cdot h_{0,T}^{-1}]$$

Then from $d^{2m}/ds^{2m} \mid_{s=0} (h_{s,T}^{-1/2} F_{s,T} h_{s,T}^{1/2} \wedge \omega_S) = h_{0,T}^{-1/2} F_{0,T}^{(2m)} h_{0,T}^{1/2} \wedge \omega_S$, we see

$$\begin{aligned} \mathbf{r}_i^{(2m)}(0,T) \quad = \quad & -C_{2m}^m \int_S [(A_0^{(m)} - A_0^{*(m)}) \wedge (A_0^{(m)} - A_0^{*(m)})]_i \\ & -C_{2m}^m \int_S [[A_0^{(m)}, [A_0^{*(m)}, h_{0,T}] h_{0,T}^{-1}]]_i \\ & -C_{2m}^m \int_S [[A_0^{(m)}, D_0' h_{0,T}^{(m)} \cdot h_{0,T}^{-1}]]_i \end{aligned}$$

The last term is zero because $D_0''^* A_0^{(m)} = 0$ and Lemma 2.3 in paper [15]. Using

$$A_0^{(m)} = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \quad \text{and} \quad H_{0,T} = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_1^{-1/2} T_2^{-1/2} I_3 \end{pmatrix}$$

one computes

$$[[A_0^{(m)}, [A_0^{*(m)}, h_{0,T}]h_{0,T}^{-1}]]_1 = (1 - T_1^{-3/2}T_2^{-1/2})C_{13}\wedge C_{13}^* + (1 - T_1^{3/2}T_2^{1/2})C_{31}^*\wedge C_{31}$$

and

$$-[(A_0^{(m)} - A_0^{*(m)})\wedge(A_0^{(m)} - A_0^{*(m)})]_1 = C_{13}\wedge C_{13}^* + C_{31}^*\wedge C_{31}$$

Therefore

$$\mathbf{r}_1^{(2m)}(0,T) = -C_{2m}^m\int_S (T_1^{-3/2}T_2^{-1/2}C_{13}\wedge C_{13}^* + T_1^{3/2}T_2^{1/2}C_{31}^*\wedge C_{31})\wedge\omega_S$$

As the same reason, we have

$$\mathbf{r}_2^{(2m)}(0,T) = -C_{2m}^m\int_S (T_1^{-1/2}T_2^{-3/2}C_{23}\wedge C_{23}^* + T_1^{1/2}T_2^{3/2}C_{32}^*\wedge C_{32})\wedge\omega_S$$

Let $B_{i3} = \sqrt{-1}\int_S C_{i3}\wedge C_{i3}^*\wedge\omega_S$ and $B_{3i} = -\sqrt{-1}\int_S C_{3i}^*\wedge C_{3i}\wedge\omega_S$ for $i = 1,2$. By our assumption, we have $B_{i3} > 0$ and $B_{3i} > 0$. Then

$$\begin{aligned}
\mathbf{r}_1^{(2m)}(0,T) &= \sqrt{-1}C_{2m}^m\{B_{13}T_1^{-3/2}T_2^{-1/2} - B_{31}T_1^{3/2}T_2^{1/2}\}\\
\mathbf{r}_2^{(2m)}(0,T) &= \sqrt{-1}C_{2m}^m\{B_{23}T_1^{-1/2}T_2^{-3/2} - B_{32}T_1^{1/2}T_2^{3/2}\}
\end{aligned}$$

Clearly, $\mathbf{r}_i^{(2m)}(0,T_0) = 0$ if

$$T_0 = (T_{01}, T_{02}) = \left(\left(\frac{B_{13}}{B_{31}}\right)^{3/8}\left(\frac{B_{23}}{B_{32}}\right)^{-1/8}, \left(\frac{B_{13}}{B_{31}}\right)^{-1/8}\left(\frac{B_{23}}{B_{32}}\right)^{3/8}\right) \tag{5.14}$$

Then we define the map $G: B_\epsilon(T_0)\to S^1(1)$ by

$$G(T) = \frac{(\mathbf{r}_1^{2m}(0,T), \mathbf{r}_2^{2m}(0,T))}{\|\,(\mathbf{r}_1^{2m}(0,T), \mathbf{r}_2^{2m}(0,T))\,\|}$$

or

$$G((T_1,T_2)) = \frac{(B_{13}T_1^{-1} - B_{31}T_1^2T_2^1, B_{23}T_2^{-1} - B_{32}T_1^1T_2^2)}{\|\,(B_{13}T_1^{-1} - B_{31}T_1^2T_2^1, B_{23}T_2^{-1} - B_{32}T_1^1T_2^2)\,\|} \tag{5.15}$$

Then it is easily proven that for ϵ small enough (here $\epsilon < 1$), G is homotopic to

$$G_1(T) = \frac{(B_{13}T_1^{-1}, B_{23}T_2^{-1})}{\|\,(B_{13}T_1^{-1}, B_{23}T_2^{-1})\,\|} = \frac{(B_{13}T_2, B_{23}T_1)}{\|\,(B_{13}T_2, B_{23}T_1)\,\|}$$

Thus $\deg G = \deg G_1 = -1$. Then as discussion of the proof of theorem 4.3 in paper [15], we see that the map for small ϵ enough,

$$(\mathbf{r}_1, \mathbf{r}_2)(s,.): B_\epsilon(T_0)\to\mathbb{R}^2, \quad s\in(0,a')$$

attains value $0\in\mathbb{R}^2$ for all $s\in(0,a')$ in $B_\epsilon(T_0)$. So for sufficiently small s, there are solution $(h_s, e^\phi\omega_S)$ to $\mathbf{L_s} = 0$ near (h_{T_0}, ω_S). From our definition of $\mathbf{L_{s,1}}$, we know that $h_{s,T}$ is the hermitian-yang-mills solution on (E, D_s). From the proposition 13 for $\mathbf{L_s}$, we have gotten the irreducible solution of stromingers system on non-Kahler manifold X. Actually we have proven

Theorem 16. *Let (E, H_0, S, ω_S) be as before. Fix its holomorphic structure D_0''. Then there is a smooth deformation D_s'' of (E, D_0'') so that there are hermitia-yang-mills metric H_s on (E, D_s) and smooth function ϕ_s on S such that*

$$\left(V = \pi^*E, \pi^*D_s'', \pi^*H_s, \tilde\omega_s = \pi^*(e^{u+\phi_s}\omega_S) + \frac{\sqrt{-1}}{2}\theta\wedge\bar\theta\right)$$

are the irreducible solutions to strominger's system on X and so that $\lim_{s\to 0}\phi_s = 0$ and $\lim_{s\to 0} H_s$ is a regular reducible hermitian Yang-Mills connection on $E = L_1 \oplus L_2 \oplus TS$.

6. Openness

We should solve the following equation

$$\sqrt{-1}\partial\bar\partial e^u \wedge \omega_S + \partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1})) + \partial\bar\partial u \wedge \partial\bar\partial u = 0 \tag{6.1}$$

by the continuity method. More precisely we introduce a parameter $t \in [0,1]$ into the equation and consider the following equation with parameter

$$\sqrt{-1}\partial\bar\partial e^u \wedge \omega_S + \mathbf{t}\partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1})) + \partial\bar\partial u \wedge \partial\bar\partial u = 0 \tag{6.2}$$

We need the following

$$\text{Elliptic condition}: \quad e^u\omega_S + \sqrt{-1}\mathbf{t}e^{-u}\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar\partial u > 0, \tag{6.3}$$

and

$$\text{Normalization}: \quad \int_S e^{-u} = A, \qquad \int_S 1 = 1 \tag{6.4}$$

So we consider the solution in the following space

$$B = \{u \in C^{k+2,\alpha} \mid u \ \text{ satisfies the conditions (6.3) and (6.4)}\} \tag{6.5}$$

Let $C_0^{k,\alpha} = \{\psi \in C^{k,\alpha} \mid \int \psi = 0\}$. Let

$$\mathbf{T} = \{t \in [0,1] \mid \text{for } \ t \ \text{ the equation (6.2) admits a solution}\}$$

Now $0 \in \mathbf{T}$ with the solution $u = -\ln A$. In this section we prove

Lemma 17. $\mathbf{T}$ *is open.*

Proof. Let $t_0 \in \mathbf{T}$ and $u(t_0)$ is the solution of the equation (6.2). Define the linear operator L from $C^{k+2,\alpha}$ to $C^{k,\alpha}$:

$$L(\phi) = *_{\omega_S}(\sqrt{-1}\partial\bar\partial(e^{u_{t_0}}\phi) \wedge \omega_S - \mathbf{t_0}\partial\bar\partial(e^{-u_{t_0}}\phi\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1})) + 2\partial\bar\partial u_{t_0} \wedge \partial\bar\partial\phi). \tag{6.6}$$

The principle part of operator $*_{\omega_S} L$ is

$$\sqrt{-1}\partial\bar\partial\phi \wedge (e^{u_{t_0}}\omega_S + \sqrt{-1}\mathbf{t_0}e^{-u_{t_0}}\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar\partial u_{t_0}).$$

From elliptic condition (6.3), we get:

$$\omega'_{t_0} = e^{u_{t_0}}\omega_S + \sqrt{-1}\mathbf{t_0}e^{-u_{t_0}}\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar\partial u_{t_0} > 0 \tag{6.7}$$

So L is the linear elliptic operator. Now Consider the operator G mapping u in B near u_{t_0} to $C_0^{k,\alpha}$:

$$*_{\omega_S}(\sqrt{-1}\partial\bar\partial e^u \wedge \omega_S + \mathbf{t}\partial\bar\partial(e^{-u}\mathrm{tr}(\bar\partial A \wedge \partial A^* \cdot g^{-1})) + \partial\bar\partial u \wedge \partial\bar\partial u).$$

Then the differential dG of G at u_{t_0} evaluated at the derivation ϕ is $L(\phi)$[4]. So $dG = L\mid_{T_{u_{t_0}}B}$, where $T_{u_{t_0}}B = \{\phi \in C^{k+2,\alpha} \mid \int e^{-u_{t_0}}\phi = 0\}$ is the tangent space of B at u_0. Before proving dG is invertible, we introduce the following elliptic operator P (ref. P.224-227 in [14]). Because ω'_{t_0} is real and positive, ω'_{t_0} can be taken as the hermitian (not Kahler !) metric on S. Let

$$P = \sqrt{-1}\Lambda_{\omega'_{t_0}}\partial\bar\partial \tag{6.8}$$

[4]We have the formula: $-8\frac{\det(u_{i\bar j})}{\det(g_{i\bar j})} = *\partial\bar\partial u \wedge \partial\bar\partial u$.

Then P is the elliptic operator on S. As the operator $\triangle$ of Kahler metric, it satisfies

$$\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega'_{t_0} = P(\psi)\frac{\omega'^2_{t_0}}{2!} \tag{6.9}$$

for any smooth function ψ on S. Furthermore, operator P and its adjoint operator P^* have the following properties:

Lemma 18. *(ref.* [14]*) 1.* $\ker(P) = \mathbb{C}$*;*
2. $\dim\ker(P^*) = 1$*, and every function* $\phi \in \ker(P^* \mid_{C^\infty(S,\mathbb{R})})$ *has constant sign.*
3. $C^\infty(S,\mathbb{R}) = \mathrm{Im}(P \mid_{C^\infty(S,\mathbb{R})}) \oplus \mathbb{R}$

Certainly, when k is big enough, the operator P acting on $C^{k,\alpha}$ and P^* acting on $C^{k+2,\alpha}$ also have the above properties. Now from the definitions of operators L, L^* and P, for any $\psi \in C^{k,\alpha}(S,\mathbb{R})$,

$$\begin{aligned}
& \int L^*(\psi)\phi\frac{\omega_S^2}{2!} \\
= & \int \psi \cdot L(\phi)\frac{\omega_S^2}{2!} \\
= & \int \psi \cdot \{\sqrt{-1}\partial\bar{\partial}(e^{u_{t_0}}\phi) \wedge \omega_S - \mathbf{t_0}\partial\bar{\partial}(e^{-u_{t_0}}\phi\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1})) + 2\partial\bar{\partial}u_{t_0} \wedge \partial\bar{\partial}\phi\} \\
= & \int \phi\sqrt{-1}\partial\partial\psi \wedge (e^{u_{t_0}}\omega_S + \sqrt{-1}\mathbf{t_0}e^{-u_{t_0}}\mathrm{tr}(\bar{\partial}A \wedge \partial A^* \cdot g^{-1}) - 2\sqrt{-1}\partial\bar{\partial}u_{t_0}) \\
= & \sqrt{-1}\int \phi\partial\bar{\partial}\psi \wedge \omega'_{t_0} \\
= & \int \phi \cdot P(\psi)\frac{\omega'^2_{t_0}}{2!} \\
= & \int P^*(\phi)\psi\frac{\omega'^2_{t_0}}{2!}
\end{aligned}$$

Then from Lemma 18, $\ker L^* = \ker P = \mathbb{R}$ and $\ker L = \ker P^* = \{\mathbb{R}\phi_0\}$, where ϕ_0 is some function has constant sign. It is clearly that $\ker L \cap T_{u_{t_0}}B = 0$. So dG is injective. Because L is linear elliptic, it is well known that the condition for $L(\phi) = \psi$ to have a weak solution on S is that $\psi \perp \ker L^*$. The Schauder theory makes sure that $\phi \in C^{k+2,\alpha}$ when $\psi \in C^{k,\alpha}$. Now for any $\psi \in C_0^{k,\alpha}$, we have $\psi \perp \ker L^*$. So there is a ϕ_1 such that $L(\phi_1) = \psi$. Take $c_0 = -\frac{\int e^{-u_{t_0}}\phi_1}{\int e^{-u_{t_0}}\phi_0}$, then $\phi_1 + c_0\phi_0 \in T_{u_{t_0}}B$ and $L(\phi_1 + c_0\phi_0) = 0$. So dG is surjective. Hence dG of G at u_{t_0} is invertible. Thus we can use the implicity function theorem to get the openness of the set $\mathbf{T}$. $\square$

7. Zero order estimate

From this section to section 10, we do estimates up to third order to equation (1.8). Let $f = \frac{1+n^2}{4} \parallel \omega_1 \parallel^2$, where ω_1 is the anti-self dual (1,1)-form on S, then the equation (1.8) is

$$\triangle\left(e^u + fe^{-u}\right) - 8\frac{\det(u_{i\bar{j}})}{\det(g_{i\bar{j}})} = 0 \tag{7.1}$$

The elliptic condition is

$$\omega' = (e^u - tfe^{-u})\omega_S - 2\sqrt{-1}\partial\bar{\partial}u > 0, \tag{7.2}$$

and the normalization is

$$\int_S e^{-u} = A, \qquad \int_S 1 = 1 \tag{7.3}$$

Timing elliptic condition $e^u - fe^{-u} > \triangle u$ by pe^{-pu}, we get

$$p(e^{-pu})(e^u - fe^{-u}) \geq p(e^{-pu}) \triangle u = - \triangle (e^{-pu}) + 4 \mid \bigtriangledown (e^{-u})^{\frac{p}{2}} \mid^2$$

Integrating, we see that

$$\int_S \mid \bigtriangledown (e^{-u})^{\frac{p}{2}} \mid^2 \leq \frac{p}{4} \int_S (e^{-u})^{p-1} \tag{7.4}$$

Applying the Sobolev inequality, we can find a constant C depending only on S such that

$$\begin{aligned} \left(\int_S (e^{-u})^{2p} \right)^{\frac{1}{2}} &\leq C \int_S (e^{-u})^p + C \int_S \mid \bigtriangledown (e^{-u})^{\frac{p}{2}} \mid^2 \\ &\leq C \int_S (e^{-u})^p + \frac{p}{4} C \int_S (e^{-u})^{p-1} \end{aligned} \tag{7.5}$$

In the following we use the constant in the generic sense. So C may mean different constants in different equations. By (7.5) and Holder inequality, we get

$$\begin{aligned} \int_S (e^{-u})^{2p} &\leq C^2 \left(\int_S (e^{-u})^p \right)^2 + C^2 p^2 \left(\int_S (e^{-u})^{p-1} \right)^2 \\ &\leq C^2 \left(\int_S (e^{-u})^p \right)^2 + C^2 p^2 \left(\int_S (e^{-u})^p \right)^{\frac{2(p-1)}{p}} \end{aligned} \tag{7.6}$$

We assume that

$$\int_S e^{-u} = A < 1 \tag{7.7}$$

We discuss the following two cases:

Case (1): For any $p \in \mathbb{Z}$, $\int_S e^{-pu} < 1$. Then $\left(\int_S (e^{-u})^p\right)^2 < \left(\int_S (e^{-u})^p\right)^{\frac{2(p-1)}{p}}$ and from (7.6),

$$\int_S (e^{-u})^{2p} \leq C^2 p^2 \left(\int_S (e^{-u})^p \right)^{\frac{2(p-1)}{p}}$$

Let $2p = 2^\beta$, then

$$\begin{aligned} \int_S (e^{-u})^{2^\beta} &\leq C^2 (2^{\beta-1})^2 \left(\int_S (e^{-u})^{2^{\beta-1}} \right)^{2(1-2^{-(\beta-1)})} \\ &\leq \left(\prod_{b=1}^{\beta-1} C^{2^b} \right) \left(\prod_{b=1}^{\beta-1} \left(2^{(\beta-b)} \right)^{2^b} \right) \left(\int_S (e^{-u})^2 \right)^{2^{\beta-1} \cdot \prod_{k=1}^{\beta-1} (1-\frac{1}{2^k})} \\ &\leq C^{2^\beta - 2} \cdot 2^{2^{\beta+1}} \left(\int_S (e^{-u})^2 \right)^{2^{\beta-1} \cdot \prod_{k=1}^{\beta-1} (1-\frac{1}{2^k})} \end{aligned}$$

where the last inequality follows by

$$\prod_{b=1}^{\beta-1} \left(2^{\beta-b} \right)^{2^b} \leq 2^{2^{\beta+1}} \tag{7.8}$$

which can be derived from following calculation:

$$
\begin{aligned}
\prod_{b=1}^{\beta-1}\left(2^{\beta-b}\right)^{2^b} &= \prod_{b=1}^{\beta-1} 2^{2^b}\prod_{b=1}^{\beta-1}\left(2^{\beta-(b+1)}\right)^{2^b} \\
&= 2^{2^\beta-2}\left(\prod_{b=1}^{\beta-1}\left(2^{\beta-(b+1)}\right)^{2^{b+1}}\right)^{\frac{1}{2}} \\
&= 2^{2^\beta-1-\beta}\left(\prod_{b=1}^{\beta-1}\left(2^{\beta-b}\right)^{2^b}\right)^{\frac{1}{2}} \\
&\leq 2^{2^\beta}\left(\prod_{b=1}^{\beta-1}\left(2^{\beta-b}\right)^{2^b}\right)^{\frac{1}{2}}
\end{aligned}
$$

So we get

$$
\left(\int(e^{-u})^{2^\beta}\right)^{\frac{1}{2^\beta}} \leq C^{1-2^{1-\beta}}\cdot 2^2\left(\int_S(e^{-u})^2\right)^{\frac{1}{2}\cdot\prod_{k=1}^{\beta-1}(1-\frac{1}{2^k})}
$$

Let $\beta\to\infty$, we see that

$$
\exp(-\inf u) = \| e^{-u} \|_\infty \leq C\left(\int_S(e^{-u})^2\right)^{\frac{B}{2}} \tag{7.9}
$$

where

$$
B = \prod_{\beta=1}^{\infty}(1-\frac{1}{2^\beta}) > 0 \tag{7.10}
$$

To finish our estimate of $\inf u$, it suffices to estimate $\| e^{-u} \|_2$. When $p=2$ the inequality (7.4) yields

$$
\int_S |\bigtriangledown(e^{-u})|^2 \leq \frac{1}{2}\int_S e^{-u} \tag{7.11}
$$

Now from normalizing condition (6.4), we have $\int_S(e^{-u}-A)=0$. So by the Poincare inequality and (7.11), we have

$$
\int_S |(e^{-u}-A)|^2 \leq C\int_S |\nabla(e^{-u}-A)|^2 \leq CA
$$

and

$$
\int_S(e^{-u})^2 \leq A^2+CA \leq CA \tag{7.12}
$$

Combining (7.9) and (7.12), we get

$$
\exp(-\inf u) = \| e^{-u} \|_\infty \leq C_1 A^{\frac{B}{2}} \tag{7.13}
$$

and

$$
\inf u \geq -\ln C_1 - \frac{B}{2}\ln A \tag{7.14}
$$

Case(2): There is a integer p such that $\int_S e^{-pu} > 1$. Let p_0 be the first such integer. Then for any $p > p_0$, by holder inequality,

$$
\int_S e^{-pu} \geq \left(\int_S e^{-p_0 u}\right)^{\frac{p}{p_0}} > 1
$$

From (7.5), we know that

$$\begin{aligned}
\int_S (e^{-u})^{2p} &\leq C^2p^2\left(\int_S (e^{-u})^p\right)^2 \qquad \text{for } p\geq p_0\\
\int_S (e^{-u})^{2p} &\leq C^2p^2\left(\int_S (e^{-u})^p\right)^{\frac{2(p-1)}{p}} \qquad \text{for } p< p_0
\end{aligned}$$

Now using above inequality, discussing as the case (1), we can get the estimate of $\inf u$. Furthermore, we still have bound (7.14) with the same B satisfy (7.10).

Next we estimate $\sup_S u$. At first, we compute

$$\begin{aligned}
&\int_S P(e^{ku})\frac{\det g'}{\det g}\frac{\omega^2}{2!}\\
&=\int_S 2g'^{i\bar{j}}\frac{\partial^2(e^{ku})}{\partial z_i\partial\bar{z}_j}\frac{\det g'}{\det g}\frac{\omega^2}{2!}\\
&=2k^2\int_S g'^{i\bar{j}}e^{ku}\frac{\partial u}{\partial z_i}\frac{\partial u}{\partial\bar{z}_j}\frac{\omega^2}{2!}+2k\int_S g'^{i\bar{j}}e^{ku}\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j}\frac{\det g'}{\det g}\frac{\omega^2}{2!}\\
&\geq k\int_S e^{ku}\left(2g'^{i\bar{j}}\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j}\frac{\det g'}{\det g}\right)\frac{\omega^2}{2!}
\end{aligned}\tag{7.15}$$

But applying the equation,

$$\begin{aligned}
&2g'^{i\bar{j}}\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j}\frac{\det g'}{\det g}\\
&=\left(2g'_{2\bar{2}}\frac{\partial^2 u}{\partial z_1\partial\bar{z}_1}+2g'_{1\bar{1}}\frac{\partial^2 u}{\partial z_2\partial\bar{z}_2}-2g'_{1\bar{2}}\frac{\partial^2 u}{\partial z_2\partial\bar{z}_1}-2g'_{2\bar{1}}\frac{\partial^2 u}{\partial z_1\partial\bar{z}_2}\right)\frac{1}{\det g}\\
&=2(e^u-fe^{-u})g^{i\bar{j}}\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j}-16\frac{\det u_{i\bar{j}}}{\det g}\\
&=(e^u-fe^{-u})(\triangle u)-2\triangle(e^u+fe^{-u})
\end{aligned}\tag{7.16}$$

Inserting (7.16) into (7.15), we get

$$\int_S P(e^{ku})\frac{\det g'}{\det g}\frac{\omega^2}{2!}\geq k\int_S e^{ku}(e^u-fe^{-u})(\triangle u)\frac{\omega^2}{2!}-2k\int_S e^{ku}\triangle(e^u+fe^{-u})\frac{\omega^2}{2!}\tag{7.17}$$

On the other hand, using the definition of operator P, we have

$$\begin{aligned}
&\int_S P(e^{ku})\frac{\det g'}{\det g}\frac{\omega^2}{2!}=\int_S P(e^{ku})\frac{\omega'^2}{2!}=\int_S\sqrt{-1}\partial\bar{\partial}(e^{ku})\wedge\omega'\\
&=\int_S\sqrt{-1}\partial\bar{\partial}(e^{ku})\wedge((e^u-fe^{-u})\omega-2\sqrt{-1}\partial\bar{\partial}u)\\
&=\int_S(e^u-fe^{-u})\triangle(e^{ku})\frac{\omega^2}{2!}+2\int_S\partial\bar{\partial}(e^{ku})\wedge\partial\bar{\partial}u\\
&=k\int_S e^{ku}(e^u-fe^{-u})(\triangle u)\frac{\omega^2}{2!}+k^2\int_S(e^u-fe^{-u})e^{ku}\mid\bigtriangledown u\mid^2\frac{\omega^2}{2!}
\end{aligned}\tag{7.18}$$

Combining (7.17) and (7.18), we see that

$$
\begin{aligned}
& k^2 \int_S (e^u - fe^{-u}) e^{ku} \mid \triangledown u \mid^2 \frac{\omega^2}{2!} \\
& \geq -2k \int_S e^{ku} \triangle (e^u + fe^{-u}) \frac{\omega^2}{2!} \\
& = -2k \int_S e^{ku} (e^u - fe^{-u}) (\triangle u) \frac{\omega^2}{2!} - 2k \int_S e^{ku} (e^u + fe^{-u}) \mid \triangledown u \mid^2 \\
& - 2k \int_S e^{(k-1)u} (\triangle f) + 4k \int_S e^{(k-1)u} \triangledown u \cdot \triangledown f
\end{aligned} \tag{7.19}
$$

Meanwhile, by integrate by part,

$$
\begin{aligned}
& - 2k \int_S e^{ku} (e^u - fe^{-u}) (\triangle u) \frac{\omega^2}{2!} \\
& = 2k(k+1) \int_S e^{(k+1)u} \mid \triangledown u \mid^2 - 2k(k-1) \int_S fe^{(k-1)u} \mid \triangledown u \mid^2 \\
& - \frac{2k}{k-1} \int_S e^{(k-1)u} \triangle f - 2k \int_S e^{(k-1)u} \triangledown u \cdot \triangledown f
\end{aligned} \tag{7.20}
$$

Inserting (7.20) into (7.19) and applying Schwarz' inequality, we get

$$
\begin{aligned}
& k^2 \int_S (e^u - fe^{-u}) e^{ku} \mid \triangledown u \mid^2 \frac{\omega^2}{2!} \\
\geq \quad & 2k^2 \int_S e^{(k+1)u} \mid \triangledown u \mid^2 - 2k^2 \int_S e^{(k-1)u} f \mid \triangledown u \mid^2 - k \int_S e^{(k-1)u} \mid \triangledown u \mid^2 \\
& -2k(1 + \frac{1}{k-1}) \int_S e^{(k-1)u} \triangle f - k \int_S e^{(k-1)u} \mid \triangledown f \mid^2
\end{aligned}
$$

and we find

$$
\begin{aligned}
& k \int_S e^{(k-1)u} \mid \triangledown f \mid^2 + 2k(1 + \frac{1}{k-1}) \int_S e^{(k-1)u} \triangle f \\
& \geq k^2 \int_S e^{(k+1)u} \mid \triangledown u \mid^2 - k^2 \int_S e^{(k-1)u} f \mid \triangledown u \mid^2 - k \int_S e^{(k-1)u} \mid \triangledown u \mid^2
\end{aligned} \tag{7.21}
$$

If we take $A > 0$ small enough such that

$$
C_1^{-1} A^{-\frac{B}{2}} > 1 + \sup f \tag{7.22}
$$

then from (7.13) we see that $\inf e^u > 1 + \sup f$ and we can estimate

$$
\begin{aligned}
& k^2 \int_S e^{(k+1)u} \mid \triangledown u \mid^2 - k^2 \int_S e^{(k-1)u} f \mid \triangledown u \mid^2 - k \int_S e^{(k-1)u} \mid \triangledown u \mid^2 \\
& \geq C k^2 \int_S e^{(k+1)u} \mid \triangledown u \mid^2 = C \frac{4k^2}{(k+1)^2} \int_S \mid \triangledown (e^u)^{\frac{k+1}{2}} \mid^2
\end{aligned} \tag{7.23}
$$

(7.21) and (7.23) imply that for all $k \geq 1$,

$$
\int_S \mid \triangledown (e^u)^{\frac{k+1}{2}} \mid^2 \leq C(k+1) \int_S e^{(k-1)u} \tag{7.24}
$$

Now applying the Sobolev inequality and since $\inf u > 0$, we get

$$\begin{aligned}\left(\int (e^u)^{2k}\right)^{\frac{1}{2}} &\leq C\int (e^u)^k + C\int \mid \triangledown (e^u)^{\frac{k}{2}} \mid^2 \\ &\leq C\int (e^u)^k + Ck\int (e^u)^{k-2} \leq Ck\int_S (e^u)^k\end{aligned} \tag{7.25}$$

Let $2k = 2^\beta$. (7.25) implies

$$\begin{aligned}\int_S (e^u)^{2^\beta} &\leq C(2^{\beta-1})^2\left(\int_S (e^u)^{2^{\beta-1}}\right)^2 \\ &\leq C^{2^\beta-2}\prod_{b=1}^{\beta-1}\left(2^{\beta-b}\right)^{2^b}\left(\int_S (e^u)^2\right)^{2^{\beta-1}} \\ &\leq C^{2^\beta-2}2^{2^{\beta+1}}\left(\int_S (e^u)^2\right)^{2^{\beta-1}}\end{aligned}$$

where the last inequality follows by (7.8). Let $\beta\to\infty$, then we get

$$\sup u = \parallel u \parallel_\infty = C\left(\int_S (e^u)^2\right)^{\frac{1}{2}} \tag{7.26}$$

So we should estimate $\parallel e^u \parallel_2$. When $k=1$, inequality (7.24) yields

$$\int_S \mid \triangledown (e^u) \mid^2 \leq C \tag{7.27}$$

Let $M_u = \int_S e^u$, then $\int_S (e^u - M_u) = 0$. Applying Poincare inequality and (7.27), we get

$$\begin{aligned}\int_S (e^u)^2 - \left(\int_S (e^u)\right)^2 &= \int_S (e^u - M_u)^2 \\ &\leq C\int_S \mid \triangledown (e^u - M_u) \mid^2 \\ &= C\int_S \mid \triangledown e^u \mid^2 \leq C\end{aligned}$$

So there is a constant C_2 depending on S, f and A (recall in (7.22)) such that

$$\int_S (e^u)^2 \leq \left(\int_S (e^u)\right)^2 + C_2 \tag{7.28}$$

Let $U_1 = \{x\in S \mid \exp(-u(x)) \geq \frac{A}{2}\}$ and $U_2 = \{x\in S \mid \exp(-u(x)) < \frac{A}{2}\}$. Then from (7.13), we have

$$\begin{aligned}A = \int_S e^{-u} &= \int_{U_1} e^{-u} + \int_{U_2} e^{-u} < \int_{U_1} e^{-\inf u} + \int_{U_2}\frac{A}{2} \\ &= e^{-\inf u}\ \mathrm{Vol}(U_1) + \frac{A}{2}(1 - \ \mathrm{Vol}(U_2)) \\ &\leq \left(C_1 A^{\frac{B}{2}} - \frac{A}{2}\right)\ \mathrm{Vol}(U_1) + \frac{A}{2}\end{aligned}$$

Because $0 < B < 1$ and $0 < A < 1$, we can choose A small enough such that

$$A < (2C_1)^{\frac{1}{1-B/2}} \tag{7.29}$$

then we have

$$\mathrm{Vol}(U_1) \geq \frac{\frac{A}{2}}{C_1 A^{\frac{B}{2}} - \frac{A}{2}} = \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1} > 0$$

and so

$$\mathrm{Vol}(U_2) = (1 - \mathrm{Vol}(U_1)) < 1 - \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1} < 1 \tag{7.30}$$

Now we want to use (7.28) and (7.30) to estimate $\| e^u \|_2$. Applying Young inequality and Holder inequality, we compute

$$\begin{aligned}
\left(\int_S e^u\right)^2 &= \left(\int_{U_1} e^u + \int_{U_2} e^u\right)^2 \\
&\leq \left(1 + \frac{1}{\epsilon}\right)\left(\int_{U_1} e^u\right)^2 + (1+\epsilon)\left(\int_{U_2} e^u\right)^2 \\
&\leq \left(1 + \frac{1}{\epsilon}\right)\left(\int_{U_1} e^{2u}\right)\mathrm{Vol}(U_1) + (1+\epsilon)\mathrm{Vol}(U_2)\left(\int_{U_2} e^{2u}\right) \\
&\leq \left(1 + \frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + (1+\epsilon)\mathrm{Vol}(U_2)\left(\int_{U_2} e^{2u}\right) \\
&\leq \left(1 + \frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + (1+\epsilon)\mathrm{Vol}(U_2)\left(\int_{S} e^{2u}\right)
\end{aligned} \tag{7.31}$$

Inserting (7.28) and (7.30) into (7.31), we have

$$\begin{aligned}
\left(\int_S e^u\right)^2 \leq \;& \left(1 + \frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + C_2(1+\epsilon)\left(1 - \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1}\right) \\
& + (1+\epsilon)\left(1 - \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1}\right)\left(\int_S e^u\right)^2
\end{aligned}$$

Taking ϵ small enough such that

$$(1+\epsilon)\left(1 - \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1}\right) < 1$$

then we get

$$\left(\int_S e^u\right)^2 < \frac{\left(1 + \frac{1}{\epsilon}\right)\left(\frac{2}{A}\right)^2 + C_2(1+\epsilon)\left(1 - \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1}\right)}{1 - (1+\epsilon)\left(1 - \frac{1}{2C_1 A^{\frac{B}{2}-1} - 1}\right)}$$

Now estimate of $\int_S (e^u)^2$ follows from (7.28) and and then estimate of $\sup u$ follows from (7.26)

8. GRADIENT ESTIMATE

Let $\ln \mid \bigtriangledown u \mid^2 + \ln v(u)$ achieves the maximum at the point $q_1 \in S$, where v is some positive function of u. Then at the point q_1 we have

$$\bigtriangledown(\mid \bigtriangledown u \mid^2) = -\left(\frac{v'(u)}{v(u)} \mid \bigtriangledown u \mid^2\right) \bigtriangledown u \tag{8.1}$$

We may choose the normal coordinate at the point q_1 such that $\frac{\partial u}{\partial z_1} \neq 0$ and $\frac{\partial u}{\partial z_2} = 0$. Actually because u is real, we can assume that $\frac{\partial u}{\partial x_1} > 0$ and $\frac{\partial u}{\partial y_1} = 0$. Thus we can assume that at the point q_1,

$$\partial_1 u \partial_{\bar{1}} u = \partial_1 u \partial_1 u = \partial_{\bar{1}} u \partial_{\bar{1}} u = \frac{1}{2} \mid \nabla u \mid^2 \tag{8.2}$$

At the point q_1, from (8.1) we can also get

$$\partial_1 \partial_1 u + \partial_1 \partial_{\bar{1}} u = \partial_{\bar{1}} \partial_{\bar{1}} u + \partial_1 \partial_{\bar{1}} u = -\frac{1}{2} \frac{v'(u)}{v(u)} \mid \nabla u \mid^2 \tag{8.3}$$

and

$$\partial_1 \partial_2 u + \partial_2 \partial_{\bar{1}} u = \partial_{\bar{1}} \partial_{\bar{2}} u + \partial_1 \partial_{\bar{2}} u = 0 \tag{8.4}$$

By the direct calculation, using (8.2), we see that at the point q_1,

$$\begin{aligned}
& P(\ln \mid \nabla u \mid^2) \mid \nabla u \mid^2 \frac{\det g'}{\det g} \\
&= 2g'^{i\bar{j}} \left\{ \frac{\partial^2 \mid \nabla u \mid^2}{\partial z_i \partial \bar{z}_j} - \frac{\partial \mid \nabla u \mid^2}{\partial z_i} \frac{\partial \mid \nabla u \mid^2}{\partial \bar{z}_j} \cdot \frac{1}{\mid \nabla u \mid^2} \right\} \frac{\det g'}{\det g} \\
&= 4g'^{i\bar{j}} (\partial_i \partial_{\bar{j}} \partial_p u \partial_{\bar{p}} u + \partial_i \partial_{\bar{j}} \partial_{\bar{p}} u \partial_p u) \frac{\det g'}{\det g} \\
&- 4g'^{i\bar{j}} (\partial_i \partial_{\bar{1}} u \partial_{\bar{1}} \partial_{\bar{j}} u + \partial_i \partial_1 u \partial_1 \partial_{\bar{j}} u) \frac{\det g'}{\det g} \\
&+ 4g'^{i\bar{j}} (\partial_i \partial_{\bar{2}} u \partial_2 \partial_{\bar{j}} u) \frac{\det g'}{\det g} + 2g'^{i\bar{j}} R^{1\bar{1}}{}_{i\bar{j}} \mid \nabla u \mid^2 \frac{\det g'}{\det g} \\
&+ 4g'^{i\bar{j}} (\partial_i \partial_2 u \partial_{\bar{2}} \partial_{\bar{j}} u) \frac{\det g'}{\det g}.
\end{aligned} \tag{8.5}$$

The last term $4g'^{i\bar{j}} (\partial_i \partial_2 u \partial_{\bar{2}} \partial_{\bar{j}} u) \frac{\det g'}{\det g} \geq 0$. So we should estimate the first four terms. By equation and (8.1), the first term in (8.5) is

$$\begin{aligned}
& 4g'^{i\bar{j}} (\partial_i \partial_{\bar{j}} \partial_p u \partial_{\bar{p}} u + \partial_i \partial_{\bar{j}} \partial_{\bar{p}} u \partial_p u) \frac{\det g'}{\det g} \\
&= 2(e^u - fe^{-u}) \nabla \triangle u \cdot \nabla u - 16 \left(\nabla \left(\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} \right) \cdot \nabla u \right) \\
&= 2(e^u - fe^{-u}) \{ \nabla \triangle u \cdot \nabla u \} - 2 \{ \nabla \triangle (e^u + fe^{-u}) \cdot \nabla u \} \\
&= -2(e^u + fe^{-u}) \triangle u \mid \nabla u \mid^2 - 2(e^u - fe^{-u}) \mid \nabla u \mid^4 \\
&- 4e^{-u} (\nabla u \cdot \nabla f) \mid \nabla u \mid^2 + 2e^{-u} \triangle f \mid \nabla u \mid^2 \\
&- 2(e^u + fe^{-u}) (\nabla \mid \nabla u \mid^2 \cdot \nabla u) + 2e^{-u} \triangle u (\nabla u \cdot \nabla f) \\
&- 2e^{-u} (\nabla u \cdot \nabla \triangle f) + 4e^{-u} (\nabla (\nabla u \cdot \nabla f) \cdot \nabla u) \\
&= -2(e^u + fe^{-u}) \triangle u \mid \nabla u \mid^2 - 2(e^u - fe^{-u}) \mid \nabla u \mid^4 \\
&- 4e^{-u} (\nabla u \cdot \nabla f) \mid \nabla u \mid^2 + 2e^{-u} \triangle f \mid \nabla u \mid^2 \\
&+ 2 \frac{v'(u)}{v(u)} (e^u + fe^{-u}) \mid \nabla u \mid^4 + 2e^{-u} \triangle u (\nabla u \cdot \nabla f) \\
&- 2e^{-u} (\nabla u \cdot \nabla \triangle f) + 4e^{-u} (\nabla (\nabla u \cdot \nabla f) \cdot \nabla u)
\end{aligned} \tag{8.6}$$

But from (8.2), (8.3) and (8.4),

$$
\begin{aligned}
&\bigtriangledown(\bigtriangledown u\cdot\bigtriangledown f)\cdot\bigtriangledown u\\
(8.7)\qquad &=\left\{-\frac{\sqrt{2}}{4}\frac{v'(u)}{v(u)}\mid\bigtriangledown u\mid(\partial_1 f+\partial_{\bar 1}f)+\frac{1}{2}(\partial_1\partial_1 f+2\partial_1\partial_{\bar 1}f+\partial_{\bar 1}\partial_{\bar 1}f)\right\}\mid\bigtriangledown u\mid^2\\
&\geq-(C_3\mid\bigtriangledown u\mid+C_3)\mid\bigtriangledown u\mid^2
\end{aligned}
$$

where in the last inequality C_3 depends on $\sup u$, $\inf u$ and f. In the following we use the constant C_3 depending on $\sup u$, $\inf u$, f and S in the generic sense. So C_3 may mean different constants in the different equations. Inserting (8.7) into (8.6) and applying Schwarz inequality, then the first term in (8.5) is

$$
\begin{aligned}
&4g'^{i\bar j}(\partial_i\partial_{\bar j}\partial_p u\partial_{\bar p}u+\partial_i\partial_{\bar j}\partial_{\bar p}u\partial_p u)\frac{\det g'}{\det g}\\
&\geq-2(e^u-fe^{-u})\mid\bigtriangledown u\mid^4+2\frac{v'(u)}{v(u)}(e^u+fe^{-u})\mid\bigtriangledown u\mid^4\\
(8.8)\qquad &\quad+2(e^u+fe^{-u})(e^u-fe^{-u}-\triangle u)\mid\bigtriangledown u\mid^2\\
&\quad-(e^u-fe^{-u}-\triangle u)(C_3\mid\bigtriangledown u\mid+C_3)\\
&\quad-C_3\mid\bigtriangledown u\mid^3-C_3\mid\bigtriangledown u\mid^2-C_3\mid\bigtriangledown u\mid-C_3
\end{aligned}
$$

Next we compute the second term in (8.5):

$$
\begin{aligned}
&-4g'^{i\bar j}(\partial_i\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar j}u+\partial_i\partial_1 u\partial_1\partial_{\bar j}u)\frac{\det g'}{\det g}\\
&=-4(e^u-fe^{-u})(\partial_i\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar i}u+\partial_i\partial_1 u\partial_1\partial_{\bar i}u)\\
(8.9)\qquad &\quad+2\times 8\frac{\det u_{i\bar j}}{\det g_{i\bar j}}(\partial_{\bar 1}\partial_{\bar 1}u+\partial_1\partial_1 u)\\
&=-4(e^u-fe^{-u})(\partial_i\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar i}u+\partial_i\partial_1 u\partial_1\partial_{\bar i}u)\\
&\quad+2(e^u-fe^{-u})(\triangle u)(\partial_{\bar 1}\partial_{\bar 1}u+\partial_1\partial_1 u)\\
&\quad+2\{(e^u+fe^{-u})\mid\bigtriangledown u\mid^2+e^{-u}\triangle f-2e^{-u}\bigtriangledown u\cdot\bigtriangledown f\}(\partial_{\bar 1}\partial_{\bar 1}u+\partial_1\partial_1 u).
\end{aligned}
$$

But the first two terms in (8.9) are equal to

$$
\begin{aligned}
&-4(e^u-fe^{-u})(\partial_i\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar i}u+\partial_i\partial_1 u\partial_1\partial_{\bar i}u)\\
&+2(e^u-fe^{-u})(\triangle u)(\partial_{\bar 1}\partial_{\bar 1}u+\partial_1\partial_1 u)\\
(8.10)\qquad &=4(e^u-fe^{-u})(\partial_{\bar 1}\partial_{\bar 1}u+\partial_1\partial_1 u)\partial_2\partial_{\bar 2}u\\
&\quad-4(e^u-fe^{-u})(\partial_2\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar 2}u+\partial_2\partial_1 u\partial_1\partial_{\bar 2}u)
\end{aligned}
$$

From (8.3) we have

$$
(8.11)\qquad \partial_1\partial_1 u+\partial_{\bar 1}\partial_{\bar 1}u=-\frac{v'(u)}{v(u)}\mid\bigtriangledown u\mid^2-2\partial_1\partial_{\bar 1}u
$$

Inserting (8.11) and (8.4) into (8.10), and using equation and Schwarz inequality, we get

$$
\begin{aligned}
&-4(e^u - fe^{-u})(\partial_i\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar i}u + \partial_i\partial_1u\partial_1\partial_{\bar i}u)\\
&+2(e^u - fe^{-u})(\triangle u)(\partial_{\bar 1}\partial_{\bar 1}u + \partial_1\partial_1 u)\\
=& -4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar 2} \mid \triangledown u\mid^2 - (e^u - fe^{-u})^2(\triangle u)\\
&-(e^u - fe^{-u})\{(e^u + fe^{-u}) \mid \triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}\\
\geq& -4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar 2}\mid\triangledown u\mid^2 + (e^u - fe^{-u})^2(e^u - fe^{-u} - \triangle u)\\
&-C_3\mid\triangledown u\mid^2 - C_3\mid\triangledown u\mid - C_3\\
\geq& -4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar 2}\mid\triangledown u\mid^2 - C_3\mid\triangledown u\mid^2 - C_3\mid\triangledown u\mid - C_3
\end{aligned}
\tag{8.12}
$$

Applying (8.11) and Schwarz inequality, the third term in (8.9) is

$$
\begin{aligned}
&2\{(e^u + fe^{-u})\mid\triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}(\partial_{\bar 1}\partial_{\bar 1}u + \partial_1\partial_1 u)\\
\geq& -2(e^u + fe^{-u})\frac{v'(u)}{v(u)}\mid\triangledown u\mid^4 - C_3\mid\triangledown u\mid^3 - C_3\mid\triangledown u\mid^2\\
&-4\{(e^u + fe^{-u})\mid\triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}u_{1\bar 1}
\end{aligned}
\tag{8.13}
$$

Inserting (8.12) and (8.13) into (8.9), we get the estimate of second term in (8.5)

$$
\begin{aligned}
&-4g'^{i\bar j}(\partial_i\partial_{\bar 1}u\partial_{\bar 1}\partial_{\bar j}u + \partial_i\partial_1u\partial_1\partial_{\bar j}u)\frac{\det g'}{\det g}\\
\geq& -2(e^u + fe^{-u})\frac{v'(u)}{v(u)}\mid\triangledown u\mid^4 - 4(e^u - fe^{-u})\frac{v'(u)}{v(u)}u_{2\bar 2}\mid\triangledown u\mid^2\\
&-4\{(e^u + fe^{-u})\mid\triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}u_{1\bar 1}\\
&-C_3\mid\triangledown u\mid^3 - C_3\mid\triangledown u\mid^2 - C_3\mid\triangledown u\mid - C_3
\end{aligned}
\tag{8.14}
$$

The third term in (8.5) is

$$
\begin{aligned}
&4g'^{i\bar j}(\partial_i\partial_{\bar 2}u\partial_2\partial_{\bar j}u)\cdot\frac{\det g'}{\det g}\\
=& -4(e^u - fe^{-u})\det(u_{i\bar j}) + 2(e^u - fe^{-u})(\triangle u)u_{2\bar 2}\\
&-2\triangle(e^u + fe^{-u})\cdot u_{2\bar 2}\\
=& -4(e^u - fe^{-u})\det(u_{i\bar j})\\
&-2\{(e^u + fe^{-u})\mid\triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}u_{2\bar 2}\\
\geq& \frac{1}{2}(e^u - fe^{-u})^2(e^u - fe^{-u} - \triangle u) - C_3\mid\triangledown u\mid - C_3\mid\triangledown u\mid^2\\
&-2\{(e^u + fe^{-u})\mid\triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}u_{2\bar 2}\\
\geq& -2\{(e^u + fe^{-u})\mid\triangledown u\mid^2 + e^{-u}\triangle f - 2e^{-u}\triangledown u\cdot\triangledown f\}u_{2\bar 2}\\
&-C_3\mid\triangledown u\mid - C_3\mid\triangledown u\mid^2
\end{aligned}
\tag{8.15}
$$

Combine following two terms in (8.14) and (8.15):

$$\begin{aligned}
& -4\{(e^u+fe^{-u}) \mid \bigtriangledown u \mid^2 + e^{-u} \bigtriangleup f - 2e^{-u} \bigtriangledown u \cdot \bigtriangledown f\} u_{1\bar{1}} \\
& -2\{(e^u+fe^{-u}) \mid \bigtriangledown u \mid^2 + e^{-u} \bigtriangleup f - 2e^{-u} \bigtriangledown u \cdot \bigtriangledown f\} u_{2\bar{2}} \\
\geq\ & \frac{1}{2}\{(e^u+fe^{-u}) \mid \bigtriangledown u \mid^2 + e^{-u} \bigtriangleup f - 2e^{-u} \bigtriangledown u \cdot \bigtriangledown f\}(e^u - fe^{-u} - 4u_{1\bar{1}}) \\
& + (e^u+fe^{-u})(e^u - fe^{-u} - \triangle u) - (e^u - fe^{-u} - \triangle u)(C_3 \mid \bigtriangledown u \mid + C_3) \\
& - C_3 \mid \bigtriangledown u \mid^3 - C_3 \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3 \\
\geq\ & (e^u+fe^{-u})(e^u - fe^{-u} - \triangle u) - (e^u - fe^{-u} - \triangle u)(C_3 \mid \bigtriangledown u \mid + C_3) \\
& - C_3 \mid \bigtriangledown u \mid^3 - C_3 \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3
\end{aligned}$$

where in the last inequality we have assumed that

$$(e^u+fe^{-u}) \mid \bigtriangledown u \mid^2 + e^{-u} \bigtriangleup f - 2e^{-u} \bigtriangledown u \cdot \bigtriangledown f \geq 0$$

Otherwise we could have gotten the estimate of $\mid \bigtriangledown u \mid^2$ at the point q_1. Then combing (8.14) and (8.15), we get

$$\begin{aligned}
& -4g'^{i\bar{j}}(\partial_i\partial_{\bar{1}}u\partial_{\bar{1}}\partial_{\bar{j}}u + \partial_i\partial_1 u\partial_1\partial_{\bar{j}}u - \partial_i\partial_{\bar{2}}u\partial_2\partial_{\bar{j}}u)\frac{\det g'}{\det g} \\
\geq\ & -2(e^u+fe^{-u})\frac{v'(u)}{v(u)} \mid \bigtriangledown u \mid^4 - 4(e^u - fe^{-u})\frac{v'(u)}{v(u)} u_{2\bar{2}} \mid \bigtriangledown u \mid^2 \\
& - C_3 \mid \bigtriangledown u \mid^3 - C_3 \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3
\end{aligned} \tag{8.16}$$

Let $R = \mathrm{Supp} \mid R^{1\bar{1}}{}_{i\bar{j}} \mid$. The forth term is

$$\begin{aligned}
& 2g'^{i\bar{j}} R^{1\bar{1}}{}_{i\bar{j}} \mid \bigtriangledown u \mid^2 \frac{\det g'}{\det g} = -8u^{i\bar{j}} R^{1\bar{1}}{}_{i\bar{j}} \mid \bigtriangledown u \mid^2 \\
\geq\ & -8R \sum_{i,j=1} \mid u_{i\bar{j}} \mid\mid \bigtriangledown u \mid^2 \geq -8R\left((\triangle u)^2 - 8\det u_{i\bar{j}}\right)^{\frac{1}{2}} \mid \bigtriangledown u \mid^2 \\
\geq\ & -16R(e^u - fe^{-u} - \triangle u) \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid^2
\end{aligned} \tag{8.17}$$

Inserting (8.8), (8.16) and (8.17) into (8.5), we can see taht

$$\begin{aligned}
& P(\ln \mid \bigtriangledown u \mid^2) \mid \bigtriangledown u \mid^2 \frac{\det g'}{\det g} \\
\geq\ & \{(3(e^u+fe^{-u}) - 16R) \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3\}(e^u - fe^{-u} - \triangle u) \\
& + \frac{v'(u)}{v(u)}(e^u - fe^{-u})(e^u - fe^{-u} - 4u_{2\bar{2}}) \mid \bigtriangledown u \mid^2 \\
& - 2(e^u - fe^{-u}) \mid \bigtriangledown u \mid^4 - C_3 \mid \bigtriangledown u \mid^3 - C_3 \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3
\end{aligned} \tag{8.18}$$

Next we compute

$$
\begin{aligned}
&P(\ln v)\frac{\det g'}{\det g}\\
&= 2g'^{i\bar{j}}\left(\frac{v'}{v}\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} + \frac{vv''-v'^2}{v^2}\frac{\partial u}{\partial z_i}\frac{\partial u}{\partial \bar{z}_j}\right)\frac{\det g'}{\det g}\\
&= \frac{v'}{v}\left\{(e^u - fe^{-u}) \triangle u - 16\frac{\det(u_{i\bar{j}})}{\det g}\right\} + \frac{vv''-v'^2}{v^2} g'^{1\bar{1}} \mid \nabla u \mid^2 \frac{\det g'}{\det g}\\
&= -\frac{v'(u)}{v(u)}(e^u - fe^{-u}) \triangle u + \frac{vv''-v'^2}{v^2} \mid \nabla u \mid^2 (e^u - fe^{-u} - 4u_{2\bar{2}})\\
&- 2\frac{v'(u)}{v(u)}\{(e^u + fe^{-u}) \mid \nabla u \mid^2 + e^{-u} \triangle f - 2e^{-u} \nabla u \cdot \nabla f\}\\
&\geq -\frac{v'(u)}{v(u)}(e^u - fe^{-u})(e^u - fe^{-u} - \triangle u) - 2\frac{v'(u)}{v(u)}(e^u + fe^{-u}) \mid \nabla u \mid^2\\
&+ \frac{vv''-v'^2}{v^2} \mid \nabla u \mid^2 (e^u - fe^{-u} - 4u_{2\bar{2}}) - C_3 \mid \nabla u \mid - C_3
\end{aligned}
\tag{8.19}
$$

From (8.18) and (8.19) we get

$$
\begin{aligned}
&P(\ln \mid \nabla u \mid^2 + \ln v) \mid \nabla u \mid^2 \frac{\det g'}{\det g}\\
&= \left\{(2(e^u + fe^{-u}) + \frac{v'(u)}{v(u)}(e^u - fe^{-u})\right\}(e^u - fe^{-u} - \triangle u) \mid \nabla u \mid^2\\
&+ \{(e^u + fe^{-u} - 16R) \mid \nabla u \mid^2 - C_3 \mid \nabla u \mid - C_3\}(e^u - fe^{-u} - \triangle u)\\
&+ \left\{\frac{vv''-v'^2}{v^2} \mid \nabla u \mid^2 + \frac{v'(u)}{v(u)}(e^u - fe^{-u})\right\}(e^u - fe^{-u} - 4u_{2\bar{2}}) \mid \nabla u \mid^2\\
&- 2\left\{(e^u - fe^{-u}) + \frac{v'(u)}{v(u)}(e^u + fe^{-u})\right\} \mid \nabla u \mid^4\\
&- C_3 \mid \nabla u \mid^3 - C_3 \mid \nabla u \mid^2 - C_3 \mid \nabla u \mid - C_3
\end{aligned}
\tag{8.20}
$$

Take

$$v(u) = e^{4\sup u - 2u} + e^{2u - 4\sup u} > 0$$

Then

$$
\begin{aligned}
v'(u) &= -2e^{4\sup u - 2u} + 2e^{2u - 4\sup u} < 0\\
v''(u) &= 4e^{4\sup u - 2u} + 4e^{2u - 4\sup u} = 4v(u) > 0
\end{aligned}
$$

So the factor first term in (8.20) is

$$2(e^u + fe^{-u}) + (e^u - fe^{-u})\frac{v'(u)}{v(u)} > 0 \tag{8.21}$$

The factor of third term in (8.20) is:

$$
\begin{aligned}
&\frac{vv''-v'2}{v^2} \mid \nabla u \mid^2 + \frac{v'}{v}(e^u - fe^{-u})\\
&= \frac{16}{v^2} \mid \nabla u \mid^2 - \frac{e^u e^{4\sup u - 2u}}{e^{4\sup u - 2u}}\\
&> \frac{16}{e^{4\sup u - 2\inf u} + 3} \mid \nabla u \mid^2 - e^{\sup u}
\end{aligned}
\tag{8.22}
$$

Choose A such that

$$C_1^{-1}A^{-\frac{B}{2}} > 7^{\frac{1}{3}} \tag{8.23}$$

then $e^{\inf u} > \frac{1}{3}\ln 7$ and the coefficient of forth term in (8.20) is

$$\begin{aligned} & -2\left\{(e^u - fe^{-u}) + \frac{v'}{v}(e^u + fe^{-u})\right\} \\ & \geq 2\frac{e^{4\sup u - u} - 3e^{3u - u\sup u}}{e^{4\sup u - 2u} + e^{2u - 4\sup u}} \\ & \geq \frac{2e^{4\sup u - u} - 6}{e^{4\sup u - u} + 1} > 1 \end{aligned} \tag{8.24}$$

Choose A such that

$$C_1^{-1}A^{-\frac{B}{2}} > 16R + 1 \tag{8.25}$$

then $e^{\inf u} > 16R + 1$. Applying all above inequalities, at last we can see that at the point q_1

$$\begin{aligned} & 0 \geq P(\ln \mid \bigtriangledown u \mid^2 + \ln v) \mid \bigtriangledown u \mid^2 \frac{detg'}{\det g} \\ & \geq (\mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3)(e^u - fe^{-u} - \triangle u) \\ & + \left\{\frac{16}{e^{4\sup u - 2\inf u} + 1} \mid \bigtriangledown u \mid^2 - e^{\sup u}\right\}(e^u - fe^{-u} - 4u_{2\bar{2}}) \mid \bigtriangledown u \mid^2 \\ & + \mid \bigtriangledown u \mid^4 - C_3 \mid \bigtriangledown u \mid^3 - C_3 \mid \bigtriangledown u \mid^2 - C_3 \mid \bigtriangledown u \mid - C_3 \end{aligned} \tag{8.26}$$

From above inequality, we can easily see that there is a constant C_4 depending on f, M, $\sup u$ and $\inf u$ such that $\mid \bigtriangledown u \mid^2 (q_1) \leq C_4$.

Since $\ln \mid \bigtriangledown u \mid^2 + \ln(e^{4\sup u - 2u} + e^{2u - \sup u})$ achieves its maximum at q_1, we get the bound of $\mid \bigtriangledown u \mid^2$:

$$\begin{aligned} \mid \bigtriangledown u \mid^2 & \leq C_4 \frac{\left(e^{4\sup u - 2u(q)} + e^{2u(q) - 4\sup u}\right)}{(e^{4\sup u - 2u} + e^{2u - 4\sup u})} \\ & \leq C_4 \frac{\left(e^{4\sup u - 2\inf u} + e^{2\inf u - 4\sup u}\right)}{(e^{2\sup u} + e^{-2\sup u})} \end{aligned} \tag{8.27}$$

9. Second order Estimate

We now do the second order a priori estimate of u. Since $(e^u - fe^{-u})g_{i\bar{j}} - 4\partial_i\partial_{\bar{j}}u$ is positive definite, to get a second order estimate of u it sufficient to have an upper bound estimate of $e^u - fe^{-u} - \triangle u$. We fix a point q_2 and choose normal coordinate at that point for $g_{i\bar{j}}$. Let $g'_{i\bar{j}} = (e^u - fe^{-u})g_{i\bar{j}} - 4\partial_i\partial_{\bar{j}}u$. We rewrite the equation as

$$\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} = F \tag{9.1}$$

where

$$F = (e^u - fe^{-u})^2 + 2\{(e^u + fe^{-u})^2 \mid \bigtriangledown u \mid^2 + e^{-u} \triangle f - 2e^{-u} \bigtriangledown u \cdot \bigtriangledown f\} \tag{9.2}$$

Differential (9.1), we have

$$g^{i\bar{j}}\frac{\partial g'_{i\bar{j}}}{\partial z_k} = g^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{1}{F}\frac{\partial F}{\partial z_k} \tag{9.3}$$

We differentiate (9.3) again and obtain

$$
\begin{aligned}
& -g'^{i\bar q} g'^{p\bar j} \frac{\partial g'_{p\bar q}}{\partial \bar z_l} \frac{\partial g'_{i\bar j}}{\partial z_k} + g'^{i\bar j} \frac{\partial^2 g'_{i\bar j}}{\partial z_k \partial \bar z_l} \\
= \quad & -g^{i\bar q} g^{p\bar j} \frac{\partial g_{p\bar q}}{\partial \bar z_l} \frac{\partial g_{i\bar j}}{\partial z_k} + g^{i\bar j} \frac{\partial^2 g_{i\bar j}}{\partial z_k \partial \bar z_l} + \frac{1}{F} \frac{\partial^2 F}{\partial z_k \partial \bar z_l} - \frac{1}{F^2} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar z_l}
\end{aligned}
$$

or

$$
\begin{aligned}
-4g'^{i\bar j} \frac{\partial^4 u}{\partial z_i \partial \bar z_j \partial z_k \partial \bar z_l} = {} & g'^{i\bar q} g'^{p\bar j} \frac{\partial g'_{p\bar q}}{\partial \bar z_l} \frac{\partial g'_{i\bar j}}{\partial z_k} - g'^{i\bar j} \frac{\partial^2 ((e^u - f e^{-u}) g_{i\bar j})}{\partial z_k \partial \bar z_l} \\
& - g^{i\bar q} g^{p\bar j} \frac{\partial g_{p\bar q}}{\partial \bar z_l} \frac{\partial g_{i\bar j}}{\partial z_k} + g^{i\bar j} \frac{\partial^2 g_{i\bar j}}{\partial z_k \partial \bar z_l} \\
& + \frac{1}{F} \frac{\partial^2 F}{\partial z_k \partial \bar z_l} - \frac{1}{F^2} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar z_l}
\end{aligned} \tag{9.4}
$$

Contracting (9.4) with g^{kl} and using the fact that the metric $g_{i\bar j}$ is Ricci-flat, we have

$$
\begin{aligned}
P(-\triangle u) = {} & g^{k\bar l} g'^{i\bar j} g'^{p\bar q} \frac{\partial g'_{i\bar q}}{\partial z_k} \frac{\partial g'_{p\bar j}}{\partial \bar z_l} - \frac{1}{2} \triangle (e^u - f e^{-u}) \sum_{i=1}^{2} g'^{i\bar i} \\
& + \frac{1}{2F} \triangle F - \frac{1}{F^2} g^{k\bar l} \frac{\partial F}{\partial z_k} \frac{\partial F}{\partial \bar z_l} + 4g'^{i\bar j} \frac{\partial^2 g^{k\bar l}}{\partial z_i \partial \bar z_j} \frac{\partial^2 u}{\partial z_k \partial \bar z_l}
\end{aligned} \tag{9.5}
$$

Timing $\frac{\det g'_{i\bar j}}{\det g_{i\bar j}}$ to above equation and using (9.3) and (9.1), we see

$$
\begin{aligned}
P(-\triangle u) \frac{\det g'_{i\bar j}}{\det g_{i\bar j}} = {} & g^{k\bar l} g'^{i\bar j} g'^{p\bar q} \left(\frac{\partial g'_{i\bar q}}{\partial z_k} \frac{\partial g'_{p\bar j}}{\partial \bar z_l} - \frac{\partial g'_{i\bar j}}{\partial z_k} \frac{\partial g'_{p\bar q}}{\partial \bar z_l} \right) \frac{\det g'_{i\bar j}}{\det g_{i\bar j}} \\
& + \frac{1}{2} \triangle F - \frac{1}{2} \triangle (e^u - f e^{-u}) \sum_{i=1}^{2} g'^{i\bar i} \frac{\det g'_{i\bar j}}{\det g_{i\bar j}} \\
& + 4g'^{i\bar j} \frac{\partial^2 g^{k\bar l}}{\partial z_i \partial \bar z_j} \frac{\partial^2 u}{\partial z_k \partial \bar z_l}
\end{aligned} \tag{9.6}
$$

Now we compute at the point q_2

$$
\begin{aligned}
& g^{k\bar{l}} g'^{i\bar{j}} g'^{p\bar{q}} \left(\frac{\partial g'_{i\bar{q}}}{\partial z_k} \frac{\partial g'_{p\bar{j}}}{\partial \bar{z}_l} - \frac{\partial g'_{i\bar{j}}}{\partial z_k} \frac{\partial g'_{p\bar{q}}}{\partial \bar{z}_l} \right) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&= g^{k\bar{l}} (g'^{i\bar{j}} g'^{p\bar{q}} - g'^{i\bar{q}} g'^{p\bar{j}}) \frac{\partial g'_{i\bar{q}}}{\partial z_k} \frac{\partial g'_{p\bar{j}}}{\partial \bar{z}_l} \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&= g^{k\bar{l}} \left(\frac{\partial g'_{2\bar{1}}}{\partial z_k} \frac{\partial g'_{1\bar{2}}}{\partial \bar{z}_l} + \frac{\partial g'_{1\bar{2}}}{\partial z_k} \frac{\partial g'_{2\bar{1}}}{\partial \bar{z}_l} - \frac{\partial g'_{1\bar{1}}}{\partial z_k} \frac{\partial g'_{2\bar{2}}}{\partial \bar{z}_l} - \frac{\partial g'_{2\bar{2}}}{\partial z_k} \frac{\partial g'_{1\bar{1}}}{\partial \bar{z}_l} \right) \\
&= 16 \sum_{i \neq j} \left(\frac{\partial^3 u}{\partial z_i \partial \bar{z}_j \partial z_k} \frac{\partial^3 u}{\partial z_j \partial \bar{z}_i \partial \bar{z}_k} - \frac{\partial^3 u}{\partial z_i \partial \bar{z}_i \partial z_k} \frac{\partial^3 u}{\partial z_j \partial \bar{z}_j \partial \bar{z}_k} \right) \\
& \quad + 4 \sum_i \frac{\partial^3 u}{\partial z_i \partial \bar{z}_i \partial z_k} \frac{\partial (e^u - f e^{-u})}{\partial \bar{z}_k} + 4 \sum_i \frac{\partial^3 u}{\partial z_i \partial \bar{z}_i \partial \bar{z}_k} \frac{\partial (e^u - f e^{-u})}{\partial z_k} \\
& \quad - 2 \frac{\partial (e^u - f e^{-u})}{\partial \bar{z}_k} \frac{\partial (e^u - f e^{-u})}{\partial z_k} \\
&= 16 \sum_{i,j} \frac{\partial^3 u}{\partial z_i \partial \bar{z}_j \partial z_k} \frac{\partial^3 u}{\partial z_j \partial \bar{z}_i \partial \bar{z}_k} - 16 \sum_{i,j} \frac{\partial^3 u}{\partial z_i \partial \bar{z}_i \partial z_k} \frac{\partial^3 u}{\partial z_j \partial \bar{z}_j \partial \bar{z}_k} \\
& \quad + 2 \bigtriangledown (\triangle u) \cdot \bigtriangledown (e^u - f e^{-u}) - | \bigtriangledown (e^u - f e^{-u}) |^2 \\
& \geq -2 | \bigtriangledown \triangle u |^2 + 2 \bigtriangledown (\triangle u) \cdot \bigtriangledown (e^u - f e^{-u}) - C_5
\end{aligned} \tag{9.7}
$$

where C_5 depends on f, M and u up to one order derivation. In the following we will use C_5 in the generic sense. Because we want to estimate the upper bound of $e^u - fe^{-u} - \triangle u$, we assume that $e^u - fe^{-u} - \triangle u$ achieves the maximum at point q_2 and we take the normal coordinate at this point for $g_{i\bar{j}}$. So at the point q_2, we have

$$
\bigtriangledown \triangle u = \bigtriangledown (e^u - f e^{-u}) \tag{9.8}
$$

Inserting (9.8) into (9.7) and then inserting (9.7) into (9.6), we obtain

$$
\begin{aligned}
P(- \triangle u) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} &\geq \frac{1}{2} \triangle F + 4 g'^{i\bar{j}} \frac{\partial^2 g^{k\bar{l}}}{\partial z_i \partial \bar{z}_j} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} \\
& \quad - \frac{1}{2} \triangle (e^u - f e^{-u}) \sum_{i=1}^{2} g'^{i\bar{i}} \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} - C_5
\end{aligned} \tag{9.9}
$$

At first, we deal with the second term in above inequality. Using equation, Schwarze inequality, and the fact metric $g_{i\bar{j}}$ is Ricci-flat, we have

$$
\begin{aligned}
4g'^{i\bar{j}}\frac{\partial^2 g^{k\bar{l}}}{\partial z_i \partial \bar{z}_j}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} &= -16\left\{\frac{\partial^2 g^{k\bar{l}}}{\partial z_1 \partial \bar{z}_1}\frac{\partial^2 u}{\partial u_2 \partial \bar{z}_2} + \frac{\partial^2 g^{k\bar{l}}}{\partial z_2 \partial \bar{z}_2}\frac{\partial^2 u}{\partial u_1 \partial \bar{z}_1}\right\}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} \\
&\quad - 16\left\{\frac{\partial^2 g^{k\bar{l}}}{\partial z_1 \partial \bar{z}_2}\frac{\partial^2 u}{\partial u_2 \partial \bar{z}_1} + \frac{\partial^2 g^{k\bar{l}}}{\partial z_2 \partial \bar{z}_1}\frac{\partial^2 u}{\partial u_1 \partial \bar{z}_2}\right\}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} \\
&\geq -64(\max R_{i\bar{j}k\bar{l}})\sum_{ij} \mid u_{i\bar{j}} \mid^2 \\
&= -16(\max R_{i\bar{j}k\bar{l}})(\triangle u)^2 - 16(\max R_{i\bar{j}k\bar{l}}) \times 8\frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} \\
&= -16(\max R_{i\bar{j}k\bar{l}})\{(\triangle u)^2 - \triangle(e^u + fe^{-u})\} \\
&\geq -16(\max R_{i\bar{j}k\bar{l}})(\triangle u)^2 - C_5 \triangle u - C_5
\end{aligned}
\tag{9.10}
$$

We also have

$$
\sum_{i=1}^{2} g'^{i\bar{i}}\frac{\det g_{i\bar{j}}}{\det g_{i\bar{j}}} = 2(e^u - fe^{-u} - \triangle u)
\tag{9.11}
$$

Inserting (9.10) and (9.11) into (9.9), we obtain

$$
\begin{aligned}
&P(-\triangle u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
\geq\ &\frac{1}{2}\triangle F - \triangle(e^u - fe^{-u})(e^u - fe^{-u} - \triangle u) \\
&- 16(\max R_{i\bar{j}k\bar{l}})(\triangle u)^2 - C_5 \triangle u - C_5 \\
\geq\ &\frac{1}{2}\triangle F + \left\{(e^u + fe^{-u}) - 16(\max R_{i\bar{j}k\bar{l}})\right\}(\triangle u)^2 - C_5 \triangle u - C_5
\end{aligned}
\tag{9.12}
$$

So we should compute

$$
\begin{aligned}
\triangle F &= \triangle(e^u - fe^{-u})^2 + 2\triangle(e^u + fe^{-u}) \mid \nabla u \mid^2 \\
&\quad + 2(e^u + fe^{-u})\triangle(\mid \nabla u \mid^2) + 2\nabla(e^u + fe^{-u}) \cdot \nabla(\mid \nabla u \mid^2) \\
&\quad + 2\triangle e^{-u}\triangle f + 2e^{-u}\triangle^2 f + 2\nabla e^{-u} \cdot \nabla \triangle f - 4\triangle e^{-u}\nabla u \cdot \nabla f \\
&\quad - 4e^{-u}\triangle(\nabla u \cdot \nabla f) - 4\nabla e^{-u} \cdot \nabla(\nabla u \cdot \nabla f) \\
&\geq +2(e^u + fe^{-u})\triangle(\mid \nabla u \mid^2) + 2\nabla(e^u + fe^{-u}) \cdot \nabla(\mid \nabla u \mid^2) \\
&\quad - 4e^{-u}\triangle(\nabla u \cdot \nabla f) - 4\nabla e^{-u} \cdot \nabla(\nabla u \cdot \nabla f) - C_5 \triangle u - C_5
\end{aligned}
\tag{9.13}
$$

Using (9.8) and the equation, we have

$$
\begin{aligned}
\triangle \mid \nabla u \mid^2 &= 4g^{k\bar{l}} \frac{\partial^2}{\partial z_k \partial \bar{z}_l}\{g_{i\bar{j}} u_i u_{\bar{j}}\} \\
&= 4g^{k\bar{k}} g^{i\bar{i}} \left\{ \frac{\partial^3 u}{\partial z_i \partial z_k \partial \bar{z}_k} \frac{\partial u}{\partial \bar{z}_i} + \frac{\partial^3 u}{\partial \bar{z}_i \partial z_k \partial \bar{z}_k} \frac{\partial u}{\partial z_i} \right\} \\
&+ 4g^{i\bar{i}} g^{k\bar{k}} \left\{ \frac{\partial^2 u}{\partial z_i \partial z_k} \frac{\partial^2 u}{\partial \bar{z}_i \partial \bar{z}_k} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_k} \frac{\partial^2 u}{\partial \bar{z}_i \partial z_k} \right\} \\
&= \nabla \triangle u \cdot \nabla u + (\triangle u)^2 - 8 \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + 4g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}} \\
&= \nabla(e^u - f e^{-u}) \cdot \nabla u + (\triangle u)^2 - \triangle(e^u + f e^{-u}) + 4g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}} \\
&\geq (\triangle u)^2 + 4\Gamma - C_5 \triangle u - C_5
\end{aligned} \tag{9.14}
$$

where we let $\Gamma = g^{i\bar{j}} g^{k\bar{l}} u_{ik} u_{\bar{j}\bar{l}}$ (see next section). Using (9.8) and schwarz inequality, we also have

$$
4e^{-u} \triangle (\nabla u \cdot \nabla f) \geq -C_5 \sum_{ij} \mid u_{i\bar{j}} \mid - C_5 \Gamma^{\frac{1}{2}} - C_5 \tag{9.15}
$$

and

$$
\begin{aligned}
&2 \nabla (e^u + f e^{-u}) \cdot \nabla(\mid \nabla u \mid^2) - 4 \nabla e^{-u} \cdot \nabla(\nabla u \cdot \nabla f) \\
&\geq -C_5 \sum_{ij} \mid u_{i\bar{j}} \mid - C_5 \Gamma^{\frac{1}{2}} - C_5
\end{aligned} \tag{9.16}
$$

Then inserting (9.14), (9.15) and (9.16) into (9.13), we see

$$
\begin{aligned}
\triangle F &\geq 2(e^u + f e^{-u})(\triangle u)^2 + 8(e^u + f e^{-u})\Gamma - C_5 \triangle u \\
&\quad - C_5 \sum_{ij} \mid u_{i\bar{j}} \mid - C_5 \Gamma^{\frac{1}{2}} - C_5 \\
&\geq 2(e^u + f e^{-u})(\triangle u)^2 - C_5 \triangle u - C_5 \sum_{ij} \mid u_{i\bar{j}} \mid - C_5
\end{aligned} \tag{9.17}
$$

Inserting (9.17) into (9.12), we obtain

$$
\begin{aligned}
P(-\triangle u) \frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} &\geq \left\{ 2(e^u + f e^{-u}) - 16(\max R_{i\bar{j}k\bar{l}}) \right\} (\triangle u)^2 \\
&\quad - C_5 \triangle u - C_5 \sum_{ij} \mid u_{i\bar{j}} \mid - C_5
\end{aligned} \tag{9.18}
$$

Next we compute

$$
\begin{aligned}
& P(e^u - fe^{-u})\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
= & 2g'^{k\bar{l}}\frac{\partial^2(e^u - fe^{-u})}{\partial z_k \partial \bar{z}_l}\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
= & \triangle(e^u - fe^{-u}) - 8\sum_{i\neq j}\{\partial_i\partial_{\bar{i}}u\partial_j\partial_{\bar{j}}(e^u - fe^{-u}) - \partial_i\partial_{\bar{j}}u\partial_j\partial_{\bar{i}}(e^u - fe^{-u})\} \\
= & \triangle(e^u - fe^{-u}) - 2(e^u + fe^{-u}) \triangle (e^u - fe^{-u}) \\
& - 8(e^{-u} - fe^{-u})\sum_{i\neq j}\{\partial_i\partial_{\bar{i}}u\partial_j u\partial_{\bar{j}}u - \partial_i\partial_{\bar{j}}u\partial_j u\partial_{\bar{i}}u\} \\
& - 8e^{-u}\sum_{i\neq j}\{\partial_i\partial_{\bar{i}}u(\partial_j u\partial_{\bar{j}}f + \partial_{\bar{j}}u\partial_j f) - \partial_i\partial_{\bar{j}}u(\partial_j u\partial_{\bar{i}}f + \partial_{\bar{i}}u\partial_j f)\} \\
& + 8e^{-u}\sum_{i\neq j}\{\partial_i\partial_{\bar{i}}u\partial_j\partial_{\bar{j}}f - \partial_i\partial_{\bar{j}}u\partial_j\partial_{\bar{i}}f\} \\
\geq & -C_5 \triangle u - C_5\sum_{ij} \mid u_{i\bar{j}} \mid - C_5
\end{aligned} \tag{9.19}
$$

Combining (9.17) and (9.19), we obtain

$$
\begin{aligned}
& P((e^u - fe^{-u} - \triangle u)\frac{\det g'_{i\bar{j}}}{\det g_{i\bar{j}}} \\
\geq & \left\{2(e^u + fe^{-u}) - 16(\max R_{i\bar{j}k\bar{l}})\right\}(\triangle u)^2 \\
& - C_5 \triangle u - C_5\sum_{ij} \mid u_{i\bar{j}} \mid - C_5 \\
\geq & (\triangle u)^2 - C_5 \triangle u - C_5
\end{aligned} \tag{9.20}
$$

because we have chosen A such that A satisfies (8.25), from which we can get $e^{\inf u} > 16R+1$. Because we assume that $e^u - fe^{-u} - \triangle u$ achieves the maximum at point q_2, then (9.20) reads as

$$
(\triangle u)^2 - C_5 \triangle u - C \leq 0 \tag{9.21}
$$

Then we can easily get the upper bound estimate of $e^u - fe^{-u} - \triangle u$.

10. THIRD ORDER ESTIMATE

Let

$$
\begin{aligned}
\Gamma &= g^{i\bar{j}}g^{k\bar{l}}u_{ik}u_{\bar{j}\bar{l}} \\
\Theta &= g'^{i\bar{r}}g'^{s\bar{j}}g'^{k\bar{t}}u_{i\bar{j}k}u_{\bar{r}s\bar{t}} \\
\Xi &= g'^{i\bar{j}}g'^{k\bar{l}}g'^{p\bar{q}}u_{ikp}u_{\bar{j}\bar{l}\bar{q}} \\
\Phi &= g'^{i\bar{j}}g'^{k\bar{l}}g'^{p\bar{q}}g'^{r\bar{s}}u_{i\bar{l}pr}u_{\bar{j}k\bar{q}\bar{s}} \\
\Psi &= g'^{i\bar{j}}g'^{k\bar{l}}g'^{p\bar{q}}g'^{r\bar{s}}u_{i\bar{l}p\bar{s}}u_{\bar{j}k\bar{q}r}
\end{aligned}
$$

We want to compute

$$
P((\kappa_1 - \triangle u)\Theta + \kappa_2(m - \triangle u)\Gamma + \kappa_3 \mid \triangledown u \mid^2 \Gamma + \kappa_4\Gamma) \tag{10.1}
$$

where all κ_i are positive constants and m is a fixed constant such that $\kappa_1 - \triangle u > 1$ and $m - \triangle u > 0$. In the following computation, when we use some basic inequalities such as Young inequality, Schwarz inequality, we will not mention it. We will use m_i to denote some positive constant which depend on f, M and u up to second order derivations and use C_6 as the constant in generic sense. We take our calculation at the point q_3 and pick the normal coordinate at this point such that $g_{i\bar{j}} = \delta_{ij}$, $\partial g_{i\bar{j}}/\partial z_k = \partial g_{i\bar{j}}/\partial \bar{z}_k = 0$. Now we compute every terms in (10.1). At first we compute

$$
\begin{aligned}
P(|\bigtriangledown u|^2) &= 4g'^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}(g^{i\bar{j}}u_i u_{\bar{j}}) \\
&= 4g'^{\alpha\bar{\beta}}g^{i\bar{j}}(u_{i\bar{\beta}\alpha}u_{\bar{j}} + u_i u_{\bar{j}\alpha\bar{\beta}} + u_{i\bar{\beta}}u_{\bar{j}\alpha} + u_{i\alpha}u_{\bar{j}\bar{\beta}}) - C_6 \\
&= 4g'^{\alpha\bar{\beta}}g^{i\bar{j}}\{u_{i\alpha}u_{\bar{j}\bar{\beta}} + u_{i\bar{\beta}\alpha}u_{\bar{j}} + u_i u_{\bar{j}\alpha\bar{\beta}} + u_{i\bar{\beta}}u_{\bar{j}\alpha}\} - C_6 \\
&\geq m_1\Gamma - C_6\Theta^{\frac{1}{2}} - C_6
\end{aligned}
\tag{10.2}
$$

Next we use equation (9.4) to compute:

$$
\begin{aligned}
P(-\triangle u) &= -4g'^{\alpha\bar{\beta}}g^{i\bar{j}}\frac{\partial^4 u}{\partial z_i\partial\bar{z}_j\partial z_\alpha\partial\bar{z}_\beta} - 4g'^{\alpha\bar{\beta}}\frac{\partial^2 g^{i\bar{j}}}{\partial z_\alpha\partial\bar{z}_j}\frac{\partial^2 u}{\partial z_i\partial\bar{z}_j} \\
&= 16g'^{\alpha\bar{p}}g'^{q\bar{\beta}}g^{i\bar{j}}u_{i\bar{\beta}\alpha}u_{\bar{j}q\bar{p}} \\
&- 4g'^{\alpha\bar{p}}g'^{q\bar{\beta}}g^{i\bar{j}}(u_{i\bar{\beta}\alpha}\partial_{\bar{j}}(e^u - fe^{-u})g_{q\bar{p}} + u_{\bar{j}q\bar{p}}\partial_i(e^u - fe^{-u})g_{\alpha\bar{\beta}}) \\
&+ F^{-1}g^{i\bar{j}}\partial_i\partial_{\bar{j}}F - F^{-2}g^{i\bar{j}}\partial_i F\partial_{\bar{j}}F - C_6 \\
&\geq m_2\Theta - C_6 g^{i\bar{j}}\partial_i\partial_{\bar{j}}(|\bigtriangledown u|^2) - C_6 g^{i\bar{j}}\partial_i(|\bigtriangledown u|^2)\partial_{\bar{j}}(|\bigtriangledown u|^2) \\
&\geq m_2\Theta - C_6\Gamma - C_6
\end{aligned}
\tag{10.3}
$$

Certainly we should also calculate:

$$
\begin{aligned}
P(\Gamma) &= 2g'^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}\Gamma \\
&= 2g'^{\alpha\bar{\beta}}g^{i\bar{j}}g^{k\bar{l}}(u_{ik\bar{\beta}\alpha}u_{\bar{j}\bar{l}} + u_{ik}u_{\bar{j}\bar{l}\alpha\bar{\beta}} + u_{ik\alpha}u_{\bar{j}\bar{l}\bar{\beta}} + u_{ik\bar{\beta}}u_{\bar{j}\bar{l}\alpha}) \\
&+ 2g'^{\alpha\bar{\beta}}\partial_\alpha\partial_{\bar{\beta}}(g^{i\bar{j}}g^{k\bar{l}})(u_{ik}u_{\bar{j}\bar{l}}) - C_6\Gamma \\
&\geq m_3\Xi + m_3\Theta - \epsilon_1\kappa_4^{-1}\Phi - C_6\kappa_4\epsilon_1^{-1}\Gamma
\end{aligned}
\tag{10.4}
$$

Combining (10.2) and (10.4), we can estimate

$$
\begin{aligned}
&P(|\bigtriangledown u|^2\,\Gamma) \\
&= P(|\bigtriangledown u|^2)\Gamma + |\bigtriangledown u|^2\,P(\Gamma) \\
&+ 2g'^{\alpha\bar{\beta}}(\partial_\alpha(|\bigtriangledown u|^2)\partial_{\bar{\beta}}\Gamma + \partial_{\bar{\beta}}(|\bigtriangledown u|^2)\partial_\alpha\Gamma) \\
&\geq m_1\Gamma^2 - C_6\Theta^{\frac{1}{2}}\Gamma - C_6\Gamma + |\bigtriangledown u|^2\,(m_3\Xi + m_3\Theta - \epsilon_1\Phi - C_6\epsilon_1^{-1}\Gamma) \\
&- C_6\Gamma^{\frac{1}{2}}(\Theta^{\frac{1}{2}} + \Xi^{\frac{1}{2}} + \Gamma^{\frac{1}{2}})\Gamma^{\frac{1}{2}} \\
&\geq m_1\Gamma^2 - \epsilon_1\kappa_3^{-1}\Phi - C_6\kappa_3\epsilon_1^{-1}\Gamma - C_6\Xi - C_6\Theta
\end{aligned}
\tag{10.5}
$$

Combining (10.3) and (10.4), we get

$$
\begin{aligned}
&P((m-\triangle u)\Gamma) \\
&= P(-\triangle u)\Gamma + (m-\triangle u)P(\Gamma) - 2g'^{\alpha\bar\beta}\{\partial_\alpha(\triangle u)\partial_{\bar\beta}\Gamma + \partial_{\bar\beta}(\triangle u)\partial_\alpha\Gamma\} \\
&\geq (m_2\Theta - C_6\Gamma - C_6)\Gamma + (m-\triangle u)(m_3\Xi + m_3\Theta - \epsilon_1\Phi - C_6\epsilon_1^{-1}\Gamma) \\
&\quad - C_6\Theta^{\frac{1}{2}}(\Theta^{\frac{1}{2}} + \Xi^{\frac{1}{2}} + \Gamma^{\frac{1}{2}})\Gamma^{\frac{1}{2}} \\
&\geq m_2\Theta\Gamma - C_6\Gamma^2 - \epsilon_3\kappa_2^{-1}\Phi - C_6\kappa_2\epsilon^{-1}\Gamma - C_6\Xi - C_6\Theta
\end{aligned}
\tag{10.6}
$$

Now we deal with

$$
\begin{aligned}
P((\kappa_1-\triangle u)\Theta) &= P(\kappa_1-\triangle u)\Theta + (\kappa_1-\triangle u)P(\Theta) \\
&\quad - 2g'^{\alpha\bar\beta}\{\partial_\alpha(\triangle u)\partial_{\bar\beta}\Theta + \partial_{\bar\beta}(\triangle u)\partial_\alpha\Theta\}
\end{aligned}
\tag{10.7}
$$

Applying (10.3), we get

$$
P(\kappa_1-\triangle u)\Theta = m_2 S^2 - C_6\Gamma\Theta - C_6\Theta \tag{10.8}
$$

Let $(\kappa_1-\triangle u)\Theta + \kappa_2(m-\triangle u)\Gamma + \kappa_3 \mid \bigtriangledown u \mid^2 \Gamma + \kappa_4\Gamma$ achieve the maximum at the point q_3. Then at the point q_3, we have,

$$
\partial_{\bar\beta}\Theta = -\frac{1}{\kappa_1-\triangle u}\{\Theta\partial_{\bar\beta}(m-\triangle u) + \kappa_2\partial_{\bar\beta}((m-\triangle u)\Gamma) + \kappa_3\partial_{\bar\beta}(\mid \bigtriangledown u\mid^2\Gamma) + \kappa_4\partial_{\bar\beta}\Gamma\}
$$

and

$$
\begin{aligned}
&g'^{\alpha\bar\beta}\{\partial_\alpha(\kappa_1-\triangle u)\partial_{\bar\beta}\Theta + \partial_{\bar\beta}(\kappa_1-\triangle u)\partial_\alpha\Theta\} \\
&= 2\mathrm{Re}\; g'^{\alpha\bar\beta}\frac{(\triangle u)_\alpha}{\kappa_1-\triangle u}\{-(\triangle u)_{\bar\beta}\Theta - \kappa_2(\triangle u)_{\bar\beta}\Gamma + \kappa_3(\mid\bigtriangledown u\mid^2)_{\bar\beta}\Gamma \\
&\qquad + [\kappa_2(m-\triangle u) + \kappa_3 \mid\bigtriangledown u\mid^2 + \kappa_4]\Gamma_{\bar\beta}\} \\
&\geq \frac{-C_6}{\kappa_1-\triangle u}\Theta^{\frac{1}{2}} \times \{\Theta^{\frac{3}{2}} + \kappa_2\Theta^{\frac{1}{2}}\Gamma + \kappa_3\Gamma^{\frac{3}{2}} \\
&\qquad + (\kappa_2+\kappa_3+\kappa_4)(\Theta^{\frac{1}{2}} + \Xi^{\frac{1}{2}} + \Gamma^{\frac{1}{2}})\Gamma^{\frac{1}{2}}\} \\
&\geq \frac{-C_6}{\kappa_1-\triangle u}\{\Theta^2 + (\kappa_2+\kappa_3+\kappa_4)(\Theta\Gamma + \Theta + \Gamma + \Xi) + \kappa_3\Gamma^2\}
\end{aligned}
\tag{10.9}
$$

At last we should estimate of $P(\Theta)$. We follow paper [23]. We can get:
(10.10)

$$
\begin{aligned}
P(\Theta) =2g'^{\alpha\bar\beta}[\quad & 2g'^{i\bar a}g'^{b\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}+2g'^{i\bar p}g'^{q\bar a}g'^{b\bar r}g'^{s\bar j}g'^{k\bar t}\\
&+2g'^{i\bar p}g'^{q\bar r}g'^{s\bar a}g'^{b\bar j}g'^{k\bar t}+2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar a}g'^{b\bar t}\\
&+\ g'^{i\bar a}g'^{b\bar r}g'^{s\bar p}g'^{q\bar j}g'^{k\bar t}+\ g'^{i\bar r}g'^{s\bar a}g'^{b\bar p}g'^{q\bar j}g'^{k\bar t}\\
&+\ g'^{i\bar r}g'^{s\bar p}g'^{q\bar a}g'^{b\bar j}g'^{k\bar t}+\ g'^{i\bar r}g'^{s\bar p}g'^{q\bar j}g'^{k\bar a}g'^{b\bar t}]\\
&\qquad\times\partial_\alpha g'_{b\bar a}\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k}u_{\bar r s\bar t}\quad\text{(first class)}\\
-2g'^{\alpha\bar\beta}&[2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}+g'^{i\bar r}g'^{s\bar p}g'^{q\bar j}g'^{k\bar t}]\\
\times&[\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k\alpha}u_{\bar r s\bar t}+\partial_\alpha g'_{q\bar p}u_{\bar r s\bar t\bar\beta}u_{i\bar j k}]\quad\text{(second class)}\\
-2g'^{\alpha\bar\beta}&[2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}+g'^{i\bar r}g'^{s\bar p}g'^{q\bar j}g'^{k\bar t}]\\
\times&[\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k}u_{\bar r s\bar t\alpha}+\partial_\alpha g'_{q\bar p}u_{i\bar j k\bar\beta}u_{\bar r s\bar t}]\quad\text{(third class)}\\
-2g'^{\alpha\bar\beta}&[2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}+g'^{i\bar r}g'^{s\bar p}g'^{q\bar j}g'^{k\bar t}]\times\partial_\alpha\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k}u_{\bar r s\bar t}\quad\text{(forth class)}\\
+2g'^{\alpha\bar\beta}&g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\times[u_{i\bar j k\bar\beta\alpha}u_{\bar r s\bar t}+u_{i\bar j k}u_{\bar r s\bar t\bar\beta\alpha}]\quad\text{(fifth class)}\\
+2g'^{\alpha\bar\beta}&g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\times[u_{i\bar j k\bar\beta}u_{\bar r s\bar t\alpha}+u_{i\bar j k\alpha}u_{\bar r s\bar t\bar\beta}]\quad\text{(sixth class)}\\
-C_6&\Theta
\end{aligned}
$$

Comparing with (A.8) in [23], we should deal with some classes in (10.10). The first class is:

$$
\begin{aligned}
&2g'^{\alpha\bar\beta}g'^{i\bar a}g'^{b\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}\partial_\alpha g'_{b\bar a}\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k}u_{\bar r s\bar t}\\
=\ &2g'^{\alpha\bar\beta}g'^{i\bar a}g'^{b\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}(-4u_{b\bar a\alpha})(-4u_{\bar p q\bar\beta})u_{i\bar j k}u_{\bar r s\bar t}\\
&-\epsilon_2(\kappa_1-\triangle u)^{-1}\Theta^2-C_6
\end{aligned}
\tag{10.11}
$$

The second class is

$$
\begin{aligned}
&-2g'^{\alpha\bar\beta}g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k\alpha}u_{\bar r s\bar t}\\
=&-4g'^{\alpha\bar\beta}\mathfrak{g}'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}\partial_{\bar\beta}((e^u-fe^{-u})g_{\bar p q}-4u_{\bar p q})u_{i\bar j k\alpha}u_{\bar r s\bar t}\\
=&-4g'^{\alpha\bar\beta}g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}(-4u_{\bar p q\bar\beta})u_{i\bar j k\alpha}u_{\bar r s\bar t}\\
&-\epsilon_1(\kappa_1-\triangle u)^{-1}\Phi-(\kappa_1-\triangle u)\epsilon_1^{-1}\Theta
\end{aligned}
\tag{10.12}
$$

As the same reason, the third class is:

$$
\begin{aligned}
&-2g'^{\alpha\bar\beta}g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}\partial_\alpha g'_{q\bar p}u_{i\bar j k\bar\beta}u_{\bar r s\bar t}\\
\geq&-2g'^{\alpha\bar\beta}g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}(-4u_{q\bar p\alpha})u_{i\bar j k\bar\beta}u_{\bar r s\bar t}\\
&-\epsilon_1(\kappa_1-\triangle u)^{-1}\Psi-(\kappa_1-\triangle u)\epsilon_1^{-1}\Theta
\end{aligned}
\tag{10.13}
$$

Next we deal with the forth class. We take the normal coordinate at the point q_3. Then according to section 1 in [23], by direct calculation, we can get

$$u_{\bar p q\bar\beta\alpha}=\partial_{\bar\beta}\partial_\alpha\partial_{\bar p}\partial_q u-u_{q\bar\gamma}R^{\bar\gamma}_{\bar p\alpha\bar\beta}=\partial_{\bar\beta}\partial_\alpha\partial_{\bar p}\partial_q u-C_6$$

So by (9.4),

$$
\begin{aligned}
& -2g'^{\alpha\bar\beta}g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}\partial_\alpha\partial_{\bar\beta}g'_{\bar p q}u_{i\bar j k}u_{\bar r s\bar t} \\
\geq & -2g'^{\alpha\bar\beta}g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}(-4u_{\bar\beta\alpha\bar p q})u_{i\bar j k}u_{\bar r s\bar t} - C_6\Theta \\
\geq & -2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\alpha\bar a}g'^{b\bar\beta}(-4u_{\bar a b\bar p})(-4u_{\alpha\bar\beta q})u_{i\bar j k}u_{\bar r s\bar t} - \epsilon\Theta^2 - C_6\Theta \\
& -2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}\left\{F^{-1}F_{q\bar p} - F^{-2}F_qF_{\bar p}\right\}u_{i\bar j k}u_{\bar r s\bar t} \\
\geq & -2g'^{i\bar p}g'^{q\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\alpha\bar a}g'^{b\bar\beta}(-4u_{\bar a b\bar p})(-4u_{\alpha\bar\beta q})u_{i\bar j k}u_{\bar r s\bar t} \\
& -C_6\Theta\Gamma - m_2/8(\kappa_1 - \triangle u)^{-1}\Theta^2 - C_6(\kappa_1-\triangle u)\Theta
\end{aligned} \tag{10.14}
$$

Now we deal with the fifth term. By direct calculation, we have

$$
\begin{aligned}
u_{i\bar j k\bar\beta\alpha} &= \partial_\alpha\partial_{\bar\beta}\partial_k\partial_{\bar j}\partial_i u + u_{p\bar j\bar\beta}R^p_{ik\alpha} + u_{p\bar j\alpha}R^p_{ik\bar\beta} + u_{i\bar p k}R^p_{j\alpha\beta} \\
&\quad + u_{p\bar j}\partial_\alpha\partial_{\bar\beta}(g^{p\bar q}\partial_k(g_{i\bar q})) \\
&= \partial_\alpha\partial_{\bar\beta}\partial_k\partial_{\bar j}\partial_i u + u_{p\bar j\bar\beta}R^p_{ik\alpha} + u_{p\bar j\alpha}R^p_{ik\bar\beta} + u_{i\bar p k}R^p_{j\alpha\beta} + C_6
\end{aligned}
$$

Differentiating (9.4) and using above equality, we can deal with the fifth class:

$$
\begin{aligned}
& g'^{\alpha\bar\beta}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}u_{i\bar j k\bar\beta\alpha}u_{\bar r s\bar t} \\
=& g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\left\{g'^{\alpha\bar p}g'^{q\bar\beta}g'_{q\bar p k}u_{\alpha\bar\beta i\bar j} - 1/4(g'^{\alpha\bar p}g'^{q\bar\beta}g'_{\bar p q\bar j}g'_{\alpha\bar\beta,i})_k\right\}u_{\bar r s\bar t} \\
& + \frac{1}{4}g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}\{-F^{-1}F_{i\bar j k} + F^{-2}(F_kF_{i\bar j} + F_iF_{\bar j k} + F_{\bar j}F_{ik}) \\
& \qquad\qquad - F^{-3}F_iF_{\bar j}F_k\}u_{\bar r s\bar t} + C_6\Theta \\
=& g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\alpha\bar\beta}(-4u_{q\bar p k})u_{\alpha\bar\beta i\bar j}u_{\bar r s\bar t} + C_6\Theta \\
& + g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}g'^{\alpha\bar p}g'^{q\bar\beta}\{(-4u_{\alpha\bar\beta k})u_{\bar p q\bar j k} + (-4u_{\bar p q\bar j})u_{\alpha\bar\beta i k})\}u_{\bar r s\bar t} \\
& - g'^{i\bar r}g'^{s\bar j}g'^{k\bar t}(g'^{\alpha\bar a}g'^{b\bar p}g'^{q\bar\beta} + g'^{\alpha\bar p}g'^{q\bar a}g'^{b\bar\beta})(-4u_{b\bar a k})(-4u_{\bar p q\bar j})u_{\alpha\bar\beta i}u_{\bar r s\bar t} \\
& - \epsilon_1(\kappa_1-\triangle u)^{-1}(\Phi+\Psi) - C_6\epsilon_1^{-1}(\kappa_1-\triangle u)\Theta - C_6\Theta\Gamma - C_6\Gamma^2
\end{aligned} \tag{10.15}
$$

Inserting (10.11)-(10.15) into (10.10), diagonalizing and simplifying, then comparing to (A.8) and (A.9) in [23], we obtain

$$
\begin{aligned}
P(\Theta) \geq & \sum g'^{i\bar{i}} g'^{j\bar{j}} g'^{k\bar{k}} g'^{\alpha\bar{\alpha}} \times \mid u_{i\bar{j}k\bar{\alpha}} - 4 \sum_p u_{i\bar{p}k} u_{\bar{j}p\bar{\alpha}} g'^{p\bar{p}} \mid^2 \\
& + \sum g'^{i\bar{i}} g'^{j\bar{j}} g'^{k\bar{k}} g'^{\alpha\bar{\alpha}} \times \mid u_{i\bar{j}k\alpha} - 4 \sum_p (u_{i\bar{p}\alpha} u_{p\bar{j}k} + u_{i\bar{p}k} u_{p\bar{j}\alpha}) g'^{p\bar{p}} \mid^2 \\
& - \frac{1}{\kappa_1 - \triangle u} \left\{ 2\epsilon_1 \Phi + 2\epsilon_1 \Psi + (\epsilon^2 + \frac{m_2}{4}) \Theta^2 \right\} \\
& - C_6 \Theta \Gamma - C_6 \Gamma^2 - C_6 \kappa_1 \epsilon_1^{-1} \Theta - C_6 \\
= & \sum g'^{i\bar{i}} g'^{j\bar{j}} g'^{k\bar{k}} g'^{\alpha\bar{\alpha}} \times \mid \sqrt{1 - 2\epsilon_1 (\kappa_1 - \triangle u)^{-1}} u_{i\bar{j}k\bar{\alpha}} \\
& \qquad - 4 \left(\sqrt{1 - 2\epsilon_1 (\kappa_1 - \triangle u)^{-1}} \right)^{-1} \sum_p u_{i\bar{p}k} u_{\bar{j}p\bar{\alpha}} g'^{p\bar{p}} \mid^2 \\
& + \sum g'^{i\bar{i}} g'^{j\bar{j}} g'^{k\bar{k}} g'^{\alpha\bar{\alpha}} \times \mid \sqrt{1 - 5\epsilon_1 (\kappa_1 - \triangle u)^{-1}} u_{i\bar{j}k\alpha} \\
& \qquad - 4 \left(\sqrt{1 - 5\epsilon_1 (\kappa_1 - \triangle u)^{-1}} \right)^{-1} \sum_p (u_{i\bar{p}\alpha} u_{p\bar{j}k} + u_{i\bar{p}k} u_{p\bar{j}\alpha}) g'^{p\bar{p}} \mid^2 \\
& + \frac{3\epsilon_1}{\kappa_1 - \triangle u} \Phi - \frac{m_2/4 + \epsilon_2 + C_6 \epsilon_1}{\kappa_1 - \triangle u} \Theta^2 - C_6 \Theta \Gamma - C_6 \Gamma - C_6 \frac{\kappa_1 - \triangle u}{\epsilon_1} \Theta \\
\geq & \frac{3\epsilon_1}{\kappa_1 - \triangle u} \Phi - \frac{m_2/4 + \epsilon_2 + C_6 \epsilon_1}{\kappa_1 - \triangle u} \Theta^2 - C_6 \Theta \Gamma - C_6 \Gamma - C_6 \frac{\kappa_1 - \triangle u}{\epsilon_1} \Theta
\end{aligned}
\tag{10.16}
$$

Inserting (10.8), (10.9) and (10.16) into (10.7), and then inserting (10.7), (10.4), (10.5) and (10.6) into (10.1), at last we obtain

$$
\begin{aligned}
& P((\kappa_1 - \triangle u)\Theta + \kappa_2 (m - \triangle u) \Gamma + \kappa_3 \mid \bigtriangledown u \mid^2 T + \kappa_4 \Gamma) \\
\geq & \left(m_2 - \frac{C_6}{\kappa_1 - \triangle u} - \frac{m^2}{4} - \epsilon_2 - C_6 \epsilon_1 \right) \Theta^2 \\
+ & \left(m_2 \kappa_2 - C_6 (\kappa_1 - \triangle u) - \frac{C_6}{\kappa_1 - \triangle u} (\kappa_2 + \kappa_3 + \kappa_4) - C_6 \right) \Theta \Gamma \\
+ & \left(m_1 \kappa_3 - \frac{C_6}{\kappa_1 - \triangle u} \kappa_3 - C_6 \kappa_2 \right) \Gamma^2 \\
+ & \left(m_3 \kappa_4 - \frac{C_6}{\kappa_1 - \triangle u} (\kappa_2 + \kappa_3 + \kappa_4) - C_6 (\kappa_2 + \kappa_3) \right) \Xi \\
- & C_6 (\kappa_i, \epsilon_1) (\Theta + \Gamma)
\end{aligned}
\tag{10.17}
$$

Now we can think the generic constant C is fixed, because we can take the biggest one. Fix ϵ_1 and ϵ_2 such that $\epsilon_2 + C_6 \epsilon_1 = \frac{m_2}{4}$. Take κ_1 big enough such that $\frac{C_6}{\kappa_1 - \triangle u} < \frac{m_2}{4}$, then

$$
\left(m_2 - \frac{C_6}{\kappa_1 - \triangle u} - \frac{m^2}{4} - \epsilon_2 - C_6 \epsilon_1 \right) \Theta^2 > \frac{m_2}{4} \Theta^2 \tag{10.18}
$$

Let

$$
k_i = \frac{\kappa_i}{\kappa_1 - \triangle u} \quad \text{for} \quad i = 2, 3, 4.
$$

We choose k_2, k_3 and k_4 such that

$$
k_2 > \frac{C_6}{m_2} + 1
$$

$$k_3 > \left(\frac{C_6}{m_1} + 1\right) k_2$$

and

$$k_4 > C_6 \frac{k_2 + k_3}{m_3} + 1.$$

Then if we choose κ_1 big enough, we have

$$\begin{aligned}(10.19) \qquad & \left(m_2\kappa_2 - C_6(\kappa_1 - \triangle u) - \frac{C_6}{\kappa_1 - \triangle u}(\kappa_2 + \kappa_3 + \kappa_4) - C_6\right)\Theta\Gamma \\ & > (m_2(\kappa_1 - \triangle u) - C_6(k_2 + k_3 + k_4) - C_6)\Theta\Gamma > \frac{m_2}{2}\kappa_1\Theta\Gamma,\end{aligned}$$

$$\begin{aligned}(10.20) \qquad & \left(m_1\kappa_3 - \frac{C_6}{\kappa_1 - \triangle u}\kappa_3 - C_6\kappa_2\right)\Gamma^2 \\ & > (m_1 k_2(\kappa_1 - \triangle u) - C_6 k_3)\Gamma^2 > \frac{m_1}{2}k_2\kappa_1\Gamma^2\end{aligned}$$

and

$$\begin{aligned}(10.21) \qquad & \left(m_3\kappa_4 - \frac{C_6}{\kappa_1 - \triangle u}(\kappa_2 + \kappa_3 + \kappa_4) - C_6(\kappa_2 + \kappa_3)\right)\Xi \\ & > (m_3(\kappa_1 - \triangle u) - C_6(k_1 + k_2 + k_3))\Xi > \frac{m_3}{2}\kappa_1\Xi\end{aligned}$$

Inserting (10.18), (10.19), (10.20) and (10.21), we see that

$$\begin{aligned}(10.22) \qquad & 0 \geq P((\kappa_1 - \triangle u)\Theta + \kappa_2(m - \triangle u)\Gamma + \kappa_3 \mid \bigtriangledown u \mid^2 \Gamma + \kappa_4\Gamma) \\ & \geq \frac{m_2}{4}\Theta^2 + \frac{m_2}{2}\kappa_1\Theta\Gamma + \frac{m_1}{2}k_2\kappa_1\Gamma^2 + \frac{m_3}{2}\kappa_1\Xi\end{aligned}$$

Above inequality gives an estimate of the the quantity $\sup_S \Theta$ and $\sup_S \Gamma$. This in turn gives the estimates of $u_{i\bar{j}k}$ and u_{ij} for all i, j, k.

11. Solving the equation

In conclusion, we have proved the following

Proposition 19. *Let S be a K3 surface with Calabi-Yau metric ω_S. Let u be a real-valued function in $C^4(S)$ such that $\int_S e^{-u}\frac{\omega^2}{2!} = A$ and $(e^u - fe^{-u})\omega_S - 2\sqrt{-1}\partial\bar{\partial}u$ defines another hermitian metric on S. Suppose*

$$\triangle(e^u - fe^{-u}) = \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}}.$$

If

$$(11.1) \qquad A < \min\left\{1, C_1^{-1}\left(\max\{7^{\frac{1}{3}}, (2C_1)^2, (1 + \sup f), 16(\max R_{i\bar{j}k\bar{l}} + 1)\}\right)^{-\frac{2}{B}}\right\}$$

where C_1 is a constant only depending on S, then there is a constant C_0 depending only on S, $\sup f$, $\sup \mid \bigtriangledown^l f \mid$, and A such that $\sup_S \mid u \mid < C_0$, $\sup_S \mid \bigtriangledown u \mid \leq C_0$, $\sup_S \mid u_{i\bar{j}} \mid < C_0$, $\sup_S \mid u_{i\bar{j}k} \mid < C_0$.

By above Proposition, we see that T is closed. Combining Lemma 17, we get the proof of Theorem 2.

12. The general case

Timing the elliptic condition $e^u\omega_S+\sqrt{-1}e^{-u}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g_S^{-1})-2\sqrt{-1}\partial\bar\partial u>0$ by pe^{-pu} and integrating, then we can get (7.4):

$$\begin{aligned}\int\mid\nabla e^{-\frac{p}{2}u}\mid^2\frac{\omega_S^2}{2!}&<\frac{p}{4}\int e^{-(p-1)u}\frac{\omega_S^2}{2!}+\frac{p}{4}\int e^{-(p+1)u}\frac{\sqrt{-1}}{2}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g^{-1})\wedge\omega_S\\&\leq\frac{p}{4}\int e^{-(p-1)u}\frac{\omega_S^2}{2!}\end{aligned}$$

because $\sqrt{-1}\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g^{-1})\wedge\omega_S\leq 0$. Then we can follow the discussion in section 7 to get the estimate $\inf u\geq-\ln C_1-\frac{B}{2}\ln A$. If A is small enough, we can get $\inf u>0$ big enough. So the term e^u is always control the term such as $e^{-u}\mid\mathrm{tr}(\bar\partial A\wedge\partial A^*\cdot g^{-1})\mid$ and the all estimates can be derived as the particular case.

References

[1] L. Alessandrini and G. Bassanelli, *Modifications of compact balanced manifolds*, R. Acad. Sci. Paris Sr. I Math. **320**, 12(1995), 1517–1522.

[2] B. Andreas, *On vector bundles and chiral matter in N = 1heterotic compactifications*, JHEP **011** (1999), p.9901.

[3] K.Becker, M.Becker, K.Dasgupta and P.S.Green, *Compactifications of heterotic strings on non-Kahler complex manifolds. I*, JHEP **0304**,2003,007, hep-th/0301106.

[4] K.Becker, M.Becker, P.S.Green, K.Dasgupta and E.Sharpe, *Compactifications of heterotic strings on non-Kahler complex manifolds. II*, Nucl.Phys. B, **678**,192004, hep-th/0310058.

[5] M.Becker and K.Dasgupta, *Kahler versus Non-Kahler Compactifications*, hepth-th/0312221.

[6] P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Vacuum Configurations for Superstrings*, Nuclear Physics B, **258**(1985), no.1, p.46-74.

[7] G.L.Cardoso, G.Curio, G.Dall'Agata, D. Lust, *Non-Kahler String Backgrounds and their Five Torsion Classes*, Nuclear Physics B, **652**(2003), no.1, 5-34.

[8] D.E. Diaconescu and G. Ionesei, *Spectral covers, charged matter and bundle cohomology*, J. High Energy Phys. **12**, paper 1 (1998).

[9] E. Goldestein and S. Prokushkin, *Geometric model for complex non-Kahler manifolds with SU(3) structure*, hep-th/0212307

[10] G.Grantcharov and Y.Poon, *Calabi-Yau Connections with Torsion on Toric Bundles*, math.DG/0306207.

[11] S. Gurrieri, J. Louis, A. Micu, D. Waldram, *Mirror Symmetry in Generalized Calabi-Yau Compactifications*, hep-th/0211182.

[12] P. Hořava and E. Witten, *Eleven-Dimensional Supergravity on a Manifold with Boundary*, Nucl. Phys. B, **475** (1996), p.94.

[13] S. Kachru, M. B. Schulz, P. K. Tripathy, and S. P. Trivedi, *New supersymmetric String Compactfications*, hep-th/0211182.

[14] M. Lubke, A. Teleman, *The Kobayashi-Hitchin Correspondence*, World Scientific, Singapore, New Jersey, London, Hong Kong.

[15] J.Li and S.-T.Yau, *The existence of supersymmetric string theory with torsion*, hep-th/0411136.

[16] P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. Ecole Norm. Sup. 13(1980)451-468.

[17] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), 261-295.

[18] A.Strominger, *Superstings with Torsion*, Nuclear Physics B, **274** (1986), no.2, p.253-284.

[19] A.Strominger, *Heterotic solution*, Nuclear Physics B, **343**(1990), 167-184.

[20] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39**(1986), no. S, suppl., S257–S293.

[21] R.O.Wells,Jr., *Differential Analysis on Complex Manifolds*, World Publishing Corporation, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo.

[22] E. Witten, *New Issuss in Manifolds of SU(3) holonomy*, Nuclear Physics B, **268**(1986), no. 1, p.79-112.

[23] S.-T. Yau, *On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampere equation*, I, Comm.Pure Appl. Math. 31(1978)339-411.

Institute of Mathematics, Fudan University, Shanghai 200433, China
E-mail address: majxfu@fudan.edu.cn

Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: yau@math.harvard.edu